

GENERAL STUDY OF GAS FLOW THROUGH ANY TUBE
AND PASSING THROUGH THE SPEED OF SOUND
(UNDER STEADY STATE CONDITIONS OR ANY OTHER CONDITION
WITH HEAT ADDITION AND POSSIBLY CHEMICAL REACTIONS)

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THE SPEED OF SOUND (UNDER STEADY STATE CONDITIONS OR ANY OTHER
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1. OBJECTIVE AND FRAMEWORK OF STUDY

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The flow of an elastic fluid in a tube, under steady state or variable state conditions, with or without heat addition, and with or without changes in chemical composition, constitutes a phenomenon whose practical applications (gas turbines, reaction propulsion systems, etc.) are important or soon will be, and which for this reason has been the subject of numerous theoretical studies from the time of Laval and Hugoniot until the present time (however, generally limited to simple special cases).

Even if we assume for simplicity that we are dealing with a perfect fluid with a flow that is both laminar and practically one-dimensional, the physical nature of the phenomenon was not always clear, sometimes not even the nature of the thermodynamic and dynamic fluid flow.

For example, the summary study of combustion in cylindrical tubes had already been conducted several years ago, when Colonel Barre [1] observed certain difficulties in interpretation of the full equation yielding the velocity, difficulties which D.E.F.A. asked me to explain three or four years ago, and it is only now that I have been able to precisely define [2], through simple differential equations, the nature of the thermodynamic fluid flow corresponding to this special case (however, without having succeeded in indicating at this time in an unquestionable manner what happens when the fluid attains the speed of sound, while we continue to furnish heat or induce chemical reactions).

Since then, various authors [4-28] have tackled this particular difficulty of the problem, but while certain of these appeared to have found the solution, they have arrived at this by means which do not permit us to clearly discern the detailed reasons, to precisely define the essential factors of the phenomenon or to predict their influence with a general formula.

This is why we recently decided to attempt to extend the very simple calculation method which permitted us to provide an exact estimate of the fluid flow occurring under steady state conditions and in a cylindrical tube to the general case under variable or steady state conditions in any tube with heat addition and chemical reaction.

*Numbers in the margin indicate pagination in the foreign text.

As we will see, the present report, which is the outcome of this study, permits us not only to establish by very simple means a differential equation whose interpretation completely explains what occurs at the moment of the passage through the speed of sound, but to define all the modes of fluid flow in the general case and in the principal special cases, as well as the influence of the different factors themselves on this flow: shape of the tube, chemical reactions, addition of exterior heat, etc.

2. GENERAL STUDY OF PHENOMENON THROUGH THERMODYNAMIC AND STANDARD FLUID MECHANICS PROCESSES

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2.1 BASIC HYPOTHESIS AND PRINCIPAL NOTATIONS

In order to avoid prematurely specifying the problem, we will begin by assuming that the fluid flows under variable state conditions in any type of tube and that energy addition is possible and even changes in chemical composition.

However, to avoid excessive complications in the calculations, we will make the following simplifying hypotheses:

- 2.1.1 - the tube has an unvarying shape;
- 2.1.2 - the fluid is perfect, ie. devoid of density;
- 2.1.3 - the flow is laminar, ie. devoid of turbulence;

2.1.4 - the flow is one-dimensional and occurs through planar sections, or at least such that at any instant orthogonal surfaces exist at any point within the fluid flows, such that the characteristics of velocity, pressure, density, temperature, and chemical composition, are very similar at any point of each of said orthogonal surfaces, so that they may be considered as equal without any appreciable error.

The principal notations are the following, others being defined throughout the report:

- W velocity of the fluid at any point for any trajectory
- s abscissa measured along an axis tangent to the trajectory of the point considered
- P pressure of the fluid at the same point
- ρ absolute density of the fluid at the same point
- T absolute temperature of the fluid at the same point
- R Mariotte-Gay-Lussac-Avogadro constant for one molecule

Σ	transversal cross-section of the tube along an orthogonal surface in the fluid flow with the abscissa s considered
M	mean molecular mass corresponds to the composition of the fluid in the cross-section Σ considered
μ	total mass flow through the cross-section
U	approximate internal energy for the unit of mass
H	approximate enthalpy for the unit of mass
E	chemical potential energy corresponding to the composition of the fluid in the cross-section considered
Q	quantity of heat received per unit of mass of the fluid work received per unit of mass
C	true specific heat at constant pressure at the temperature T for the unit of mass of the fluid contained in the cross-section
c	true specific heat at constant volume ratio of the true specific heats
a	"speed of sound" corresponding to the cross-section Σ considered
t	time

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2.2 DIFFERENTIAL STUDY OF FLOW UNDER VARIABLE STATE CONDITIONS

2.2.1 Fundamental Equation for Fluid Mechanics

First we will write the fundamental equation of mechanics in the form corresponding to one of the general equations for non-viscous fluid mechanics, in the case where an axis OS is tangent to the trajectory at the point considered, and where we may neglect the influence of the weight field:

$$(1) \quad \frac{dW}{dt} = \frac{\partial W}{\partial t} + W \frac{\partial W}{\partial s} = -\frac{1}{\rho} \frac{\partial P}{\partial s}$$

2.2.2 Continuity Equation

We will write the continuity equation in the classical form:

$$\frac{\partial (\rho \Sigma W)}{\partial s} + \frac{\partial (\rho \Sigma)}{\partial t} = 0$$

or, by noting that Σ is a function of s alone:

$$\frac{\partial \rho}{\partial t} = -\frac{1}{\Sigma} \frac{\partial (\rho \Sigma W)}{\partial s},$$

or:

$$\frac{\partial \rho}{\partial t} = -\left[W \frac{\partial \rho}{\partial s} + \rho \frac{\partial W}{\partial s} + \frac{1}{\Sigma} \rho W \frac{\partial \Sigma}{\partial s} \right],$$

or:

$$(2) \quad \boxed{\frac{1}{\rho} \left(\frac{\partial \rho}{\partial t} + W \frac{\partial \rho}{\partial s} \right) = \frac{1}{\rho} \frac{d\rho}{dt} = -\frac{\partial W}{\partial s} - \frac{W}{\Sigma} \frac{\partial \Sigma}{\partial s}}$$

However, obviously we also may express the continuity by writing: /5

$$\mu = \rho \Sigma W,$$

where:

$$(3) \quad \frac{d\mu}{\mu} = \frac{d\rho}{\rho} + \frac{d\Sigma}{\Sigma} + \frac{dW}{W},$$

or:

$$(3') \quad \boxed{\frac{1}{\rho} \left(\frac{\partial \rho}{\partial t} + W \frac{\partial \rho}{\partial s} \right) = -\frac{1}{W} \left(\frac{\partial W}{\partial t} + W \frac{\partial W}{\partial s} \right) - \frac{W}{\Sigma} \frac{\partial \Sigma}{\partial s} - \frac{1}{\mu} \left(\frac{\partial \mu}{\partial t} + W \frac{\partial \mu}{\partial s} \right)}$$

Which, through comparison with (2), yields:

$$(4) \quad \boxed{\frac{\partial W}{\partial t} = \frac{W}{\mu} \frac{\partial \mu}{\partial t} + \frac{W^2}{\mu} \frac{\partial \mu}{\partial s}}$$

This equation is simplified in the case where the variation of conditions is sufficiently slow such that the mass flow is practically identical through all cross-sections of the tube at a given instant, ie. in the case where $\frac{\partial \mu}{\partial s}$ is zero, yielding:

$$(4') \quad \frac{\partial W}{\partial t} = \frac{W}{\mu} \frac{\partial \mu}{\partial t}.$$

2.2.3 Conservation of Energy

Finally, the conservation of energy obviously may be written as follows, with the notations adopted:

$$d\left(\frac{1}{2} W^2\right) + dU + dE = dQ + d\zeta$$

and, by taking into consideration that $d\zeta = -d(pv)$:

$$d\left(\frac{1}{2} W^2\right) = dQ - dE - dH,$$

or:

$$W \left(\frac{\partial W}{\partial t} dt + \frac{\partial W}{\partial s} ds \right) = dQ - dE - dH.$$

or:

$$(5) \quad \boxed{W \left(\frac{\partial W}{\partial t} + W \frac{\partial W}{\partial s} \right) = \frac{dQ}{dt} - \frac{dE}{dt} - \frac{dH}{dt}}.$$

2.2.4 Fluid State Equation

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In the case where the fluid is gaseous and where we may apply the Mariotte-Gary-Lussac and Avogadro laws without appreciable error, we also may write:

$$(6) \quad \boxed{\frac{P}{\rho} = \frac{R}{M} T}.$$

where:

$$(6') \quad \boxed{\frac{dP}{P} = -\frac{dM}{M} + \frac{d\rho}{\rho} + \frac{dT}{T}}.$$

It obviously is difficult to completely resolve the question in the most general case without defining any of the conditions required at the boundaries of the fluid flow; however, several general considerations are possible, which result from the direct study of the fundamental equations above or their combinations.

2.2.5 Relationships Between Kinetic Energy and Pressure Variations

Equation (1) expresses that the fluid acceleration is always in the direction opposite to the pressure gradient existing in the cross-section considered and numerically equal to the quotient of this pressure gradient divided by the density. Although the first part of this proposition is obvious, there will be good reason for constantly remaining within the framework of this for the physical interpretation of the phenomenon.

By multiplying by ds , equation (1) obviously may be written as follows:

$$(7) \quad \boxed{W \frac{\partial W}{\partial s} ds = -\frac{1}{\rho} \frac{\partial P}{\partial s} ds - \frac{\partial W}{\partial t} ds},$$

or by integrating:

$$(7') \quad \boxed{\frac{(Ws)^2 - (Ws_0)^2}{2} = -\int_{s_0}^s \frac{dP}{\rho} - \int_{s_0}^s \frac{\partial W}{\partial t} ds}.$$

These forms (7) and (7') deduced from the general equation for fluid mechanics with tangent axes (1), demonstrate, as we have already indicated in previous publications:

-that the increase in kinetic energy produced between two cross-sections of abscissa s_0 and s under variable state conditions is equal to the increase of kinetic energy which was produced under steady state conditions between the same cross-sections, deducting the integral

$$\int_{s_0}^s \frac{\partial W}{\partial t} ds;$$

-that consequently, when the conditions are slightly variable, it is possible to obtain an approximate value for the increase in actual kinetic energy by calculating the increase in kinetic energy which was produced under steady state conditions [according to the Saint-Vincent formula, ie. by neglecting the integral $\int_{s_0}^s \frac{\partial W}{\partial t} ds$ in formula (7')], and by

deducting an approximate value for the integral $\int_{s_0}^s \frac{\partial W}{\partial t} ds$ deduced from the values of W thus calculated;

-that in conformance with equation (7'), the increase in kinetic energy produced between two cross-sections of abscissa s_0 and s may remain positive, even when the corresponding pressure difference is negative, and consequently even if motion is "decelerated", ie. if

$$-\int_{s_0}^s \frac{\partial W}{\partial t} ds$$

is sufficiently positive. In fact, this last remark constitutes the explanation for the phenomenon observed with engine exhausts which is known under the name of the Kadenacy effect, a phenomenon whose occurrence has been recognized even in the absence of any exhaust ducts.

2.2.6 Differential Equation Yielding Velocity Gradient

In equation (5), we may express $dH=d(CT)$, and assuming that C is constant, or at least neglecting $\frac{dC}{dt}$ with respect to $C\frac{dT}{dt}$, which

yields:

$$W \frac{dW}{dt} = W \left(\frac{\partial W}{\partial t} + W \frac{\partial W}{\partial s} \right) = \frac{dQ - dE}{dt} - C \frac{dT}{dt},$$

where:

$$\frac{dW}{dt} = \frac{1}{W} \left[\frac{dQ - dE}{dt} - CT \left(\frac{1}{P} \frac{dP}{dt} + \frac{1}{M} \frac{dM}{dt} - \frac{1}{\rho} \frac{d\rho}{dt} \right) \right],$$

and since:

$$CT = \frac{MP}{R\rho} C = \frac{\gamma}{\gamma-1} \frac{P}{\rho} = \frac{a^2}{\gamma-1},$$

it becomes, by considering (2):

$$(8) \quad \frac{dW}{dt} = \frac{1}{W} \left[\frac{dQ - dE}{dt} - \frac{a^2}{\gamma-1} \left(\frac{1}{P} \frac{dP}{dt} + \frac{1}{M} \frac{dM}{dt} + \frac{\partial W}{\partial s} + \frac{W}{\Sigma} \frac{\partial \Sigma}{\partial s} \right) \right].$$

But, by virtue of (1), we have:

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + W \frac{\partial P}{\partial s} = \frac{\partial P}{\partial t} - \rho W \left(\frac{\partial W}{\partial t} + W \frac{\partial W}{\partial s} \right)$$

and consequently:

$$(9) \quad \frac{a^2}{(\gamma-1)PW} \frac{dP}{dt} = \frac{a^2}{(\gamma-1)W} \frac{1}{P} \frac{\partial P}{\partial t} - \frac{\gamma}{\gamma-1} \left(\frac{\partial W}{\partial t} + W \frac{\partial W}{\partial s} \right);$$

by defining $\frac{dW}{dt}$ as a function of the partial differentials for the velocity, (8) thus may be written:

$$\begin{aligned} & \left(\frac{\partial W}{\partial t} + W \frac{\partial W}{\partial s} \right) \left(1 - \frac{\gamma}{\gamma-1} \right) \\ &= \frac{1}{W} \left[\frac{dQ-dE}{dt} \right] - \frac{a^2}{(\gamma-1)WM} \frac{dM}{dt} - \frac{a^2}{(\gamma-1)W} \frac{\partial W}{\partial s} - \frac{a^2}{\gamma-1} \frac{1}{\Sigma} \frac{\partial \Sigma}{\partial s}, \\ (10) \quad & \boxed{(a^2 - W^2) \frac{\partial W}{\partial s} = W \frac{\partial W}{\partial t} + (\gamma-1) \frac{dQ-dE}{dt} - a^2 \frac{1}{M} \frac{dM}{dt} - a^2 W \frac{1}{\Sigma} \frac{\partial \Sigma}{\partial s}} \end{aligned}$$

or even, since Σ is a function of s alone, and since consequently:

$$\begin{aligned} W \frac{\partial \Sigma}{\partial s} &= \frac{d\Sigma}{ds} \frac{ds}{dt} = \frac{d\Sigma}{dt} \\ (a^2 - W^2) \frac{\partial W}{\partial s} &= W \frac{\partial W}{\partial t} + \left[(\gamma-1) \frac{dQ-dE}{dt} - a^2 \left(\frac{1}{M} \frac{dM}{dt} + \frac{1}{\Sigma} \frac{d\Sigma}{dt} \right) \right]. \end{aligned}$$

2.2.7 Discussion of Differential Equation (10)

If we consider formula (10), we can readily extract an entire series of precise indications on the nature of the flow under the different conditions which can be anticipated. In fact, we see:

2.2.7.1 Influence of Velocity

-that when all other factors are equal (ie., for the same flow acceleration value $\frac{\partial W}{\partial t}$, energy addition $\frac{dQ-dE}{dt}$, variation of molecular mass $\frac{1}{M} \frac{dM}{dt}$, and divergence of the tube in the cross-section considered $\frac{1}{\Sigma} \frac{\partial \Sigma}{\partial s}$, the absolute value of the velocity gradient increases more and more rapidly with the value for this velocity W , as long as this remains less than the "speed of sound" a , and that on the contrary, the absolute value of the velocity gradient decreases when W increases for values of

W greater than the "speed of sound" a .

2.2.7.2 Sign of Velocity Gradient

-that the velocity gradient is positive for values of W less than a and negative for values of W greater than a , when the second term is positive (ie., when $\frac{1}{\Sigma} \frac{\partial \Sigma}{\partial s}$ is negative in the absence of energy

addition, variation of molecular mass and flow acceleration, or when the energy addition and positive flow acceleration, or the sum of the two corresponding terms and of the term $\frac{1}{M} \frac{dM}{dt}$, cancels out the term $\frac{1}{\Sigma} \frac{\partial \Sigma}{\partial s}$,

when the cross-section considered corresponds to the divergent portion of the tube).

Inversely, the velocity gradient is negative for velocities less than a , and positive for velocities greater than a when the second term is negative.

2.2.7.3 Case Steady State and Isentropic Flow for Constant Composition Fluids

-that consequently, and in the special case of a steady state flow without energy addition nor chemical reaction, the velocity gradient cannot remain positive, regardless of the value of W , and that $\frac{\partial \Sigma}{\partial s}$ is

negative for W less than a and positive for W greater than a , ie. the tube is convergent for velocities less than the "speed of sound" and divergent for velocities greater than the "speed of sound". In addition, we discover in this case the necessary existence of a "throat", for which $\frac{\partial \Sigma}{\partial s} = 0$, for $W = a$ if the flow is laminar (in fact, if

there were no throat, the velocity gradient in the corresponding cross-section would pass from $+\alpha$ to $-\alpha$, and consequently the pressure gradient would pass from $-\alpha$ to $+\alpha$ or vice versa, as (1) demonstrates; as a result, the distribution of pressures around the cross-section Σ would conform to figure 1 below and two shock waves of opposite directions would appear in this cross-section).

These waves having a velocity greater than the speed of sound cannot fail to be propagated in the upstream and downstream cross-sections and they would be capable of producing perturbations there, as a function of the gas supply or gas evacuation conditions, which in the end would result in subsequent development and possible stability of the flow.

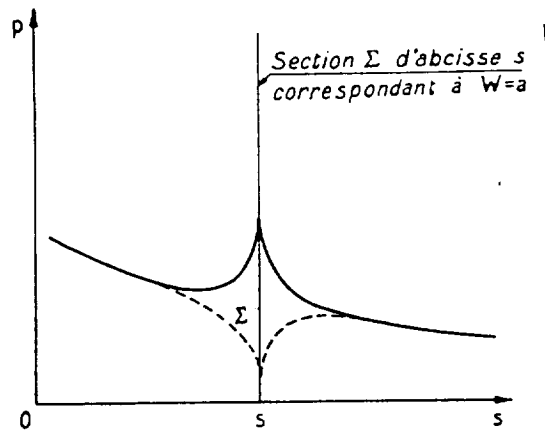


Figure 1.

----- Second term of (10) or (11) > 0 } for $W=a$
 _____ Second term of (10) or (11) < 0 }

Key: 1-Cross-section Σ of abscissa s corresponding to $W=a$.

2.2.7.4 Influence of Energy Flux

-that all other factors being equal (ie. for the same flow acceleration value $\frac{\partial W}{\partial t}$ and for the same divergence of the tube in the cross-section considered $\frac{1}{\Sigma} \frac{\partial \Sigma}{\partial s}$, and for the same value of $\frac{W}{a}$), the absolute value of the velocity gradient increases as the energy flux $\frac{dQ-dE}{dt}$ increases (disregarding the generally small influence of the corrective term $\frac{1}{M} \frac{dM}{dt}$), while the second term of equation (10) (disregarding $\frac{dQ-dE}{dt}$) is positive, and that the reverse is true in the opposite case [as long as $\frac{dQ-dE}{dt}$ does not become large enough to change the sign of the second term of (10)]. /10

-that in a more general fashion, positive energy flux $\frac{dQ-dE}{dt}$ acts on the velocity gradient in the same way as convergence of the tube, all other things being equal.

2.2.7.5 Influence of Variations in Tube Cross-Section

-that all other factors being equal (ie. for the same flow acceleration value $\frac{\partial W}{\partial t}$ and for the same energy addition $\frac{dQ-dE}{dt}$ and

for the same Mach number $\frac{W}{a}$, the velocity gradient increases as

$\frac{1}{\Sigma} \frac{\partial \Sigma}{\partial s}$ increases in absolute value [assuming that $\frac{\partial \Sigma}{\partial s}$ is of opposite

sign to the sum of the other terms of the second term of equation (10) and that we can neglect the possible variations of the term $\frac{dM}{dt}$ of

this second term] and that in a more general fashion, the pressure gradient assumes the same value as if there was no energy flux nor variation of mean molecular mass, all other factors being equal (ie., the Mach number and flow acceleration), and that the divergence of the tube has the following algebraic value:

$$\frac{1}{\Sigma} \frac{d\Sigma}{dt} + \frac{1}{M} \frac{dM}{dt} - \frac{\gamma - 1}{a^2} \frac{dQ - dE}{dt}.$$

2.2.7.6 Influence of Flow Acceleration

-that all other factors being equal (ie. for the same energy flux, same variation of mean molecular mass and same convergence or divergence of the tube), the velocity gradient, which is assumed to be positive, increases if there is acceleration of the flow and if W is less than a , and that the reverse is true if W is greater than a .

2.2.7.7 Appearance of Shock Waves for $W=a$ in General Case

-that if $W=a$ (ie., if we find ourselves in a cross-section where the fluid attains the "speed of sound") and if the second term is not zero, the velocity gradient $\frac{\partial W}{\partial s}$ passes from $+\alpha$ to $-\alpha$, which by virtue of

equation (1), produces passage of the pressure gradient $\frac{\partial p}{\partial s}$ from $-\alpha$

to $+\alpha$, and consequently, as we have seen in paragraph 2.2.7.3, the appearance of two shock waves of opposite directions having propagation speeds greater than a , which has the effect of almost immediately modifying the boundary conditions, and consequently the nature of the flow.

2.2.7.8 Possibilities of Passing Through Speed of Sound Without Flow Discontinuity

In a general fashion, we see that the velocity W cannot exceed the value a (however variable) for the "speed of sound" without producing discontinuity in the velocity or pressure gradients if the second term of equation (10) is zero, ie. if a precise relationship exists in the cross-section considered between the divergence of the tube, the energy flux, the variation of mean molecular mass and the local acceleration of motion $\frac{\partial W}{\partial t}$.

Consequently: If we are given the divergence of the tube, the energy flux and the variation of mean molecular mass in the cross-section Σ ,

where $W=a$, the velocity and pressure gradients would remain finite only if the motion is accelerated (or decelerated) and if the flow acceleration $\frac{\partial W}{\partial t}$ assumes the precise value which cancels the second term of equation (10).

On the contrary, if we are given the flow acceleration $\frac{\partial W}{\partial t}$, the energy flux and the variation of mean molecular mass, the velocity and pressure gradients may remain finite for $W=a$ only if the divergence of the tube assumes the precise value which cancels the second term of equation (10).

In addition, if we were given $\frac{\partial W}{\partial t}$, $\frac{1}{\Sigma} \frac{\partial \Sigma}{\partial s}$ and $\frac{1}{M} \frac{dM}{dt}$, the velocity and pressure gradients would remain finite for $W=a$ only if the energy flux $\frac{dQ-dE}{dt}$ assumed the precise value which would cancel the second term of equation (10) under these conditions.

2.2.7.9 Passing Through $W=a$ Under Steady State Conditions

In a general fashion, we also see that if there are steady state conditions, the "speed of sound" corresponds to the "throat" of the tube and the throat to the "speed of sound" only if the sum of the terms [of the second term of equation (10)] pertaining to the energy flux and the variation of mean molecular mass is zero.

2.2.7.10 Passing Through $W=a$ With Cylindrical Tubes

In this case, it is obvious that the velocity W cannot exceed the value a corresponding to the "speed of sound" without the velocity and pressure gradients becoming infinite (ie. without appearance of shock waves) unless the flow acceleration in the corresponding cross-section has the precise value $\frac{\partial W}{\partial t}$ which cancels the second term of equation

(10), or if the energy flux $\frac{dQ-dE}{dt}$ is cancelled and changes sign for $W=a$.

However, it should be noted that the flow acceleration required to compensate for positive heat addition, or more generally a positive value for the set of terms in $\frac{dQ-dE}{dt}$ and $\frac{dM}{dt}$, is a deceleration, ie., a slowing of the flow.

2.2.7.11 General Conclusions Concerning Possibility of Transferring Energy to Fluids Having Attained Speed of Sound

From the preceding discussions, we see that it is always possible to transfer energy to a fluid, but that for a given velocity, this is

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possible only in a certain fashion; if the flow velocity is equal to the "speed of sound", a precise relationship must exist between the divergence of the tube $\frac{1}{\Sigma} \frac{\partial \Sigma}{\partial s}$, the heat flux transferred to the fluid

$\frac{dQ}{dt}$, the energy flux $-\frac{dE}{dt}$ resulting from chemical reactions, the

variation of mean molecular mass $\frac{1}{M} \frac{dM}{dt}$, and the flow acceleration $\frac{\partial W}{\partial t}$

in the cross-section where this particular velocity is attained; lacking this, a discontinuity in the values of the velocity and pressure gradients appeared at the level of this cross-section, which pass from $+\alpha$ to $-\alpha$ or vice versa, with the consequence of the formation of two shock waves of opposite directions which very rapidly perturb the conditions at the boundaries of the flow (to the extent that they are capable of being perturbed).

2.2.7.12 Possibilities of Passing Through Speed of Sound Without Changing Sign of Velocity Gradient

It is obvious that the required condition sufficient such that the velocity always varies in the same direction, or more particularly so that its derivative does not change sign at the moment when we cross the "speed of sound", ie. in the cross-section where $W=a$, is that the second term changes sign for $W=a$, and for this velocity value alone. The convergence of the upstream portion and the divergence of the downstream portion with respect to the cross-section $W=a$, recognized to be necessary in the case of steady state and isentropic flow of a constant composition fluid, is only a special case of this proposition, which is however at least theoretically applicable to both the acceleration of the fluid from a low velocity to a supersonic velocity and to the deceleration of a fluid from a supersonic velocity to a subsonic or zero velocity.

2.2.8 Differential Equation Defining Pressure Gradient

Equation (1) obviously can be placed in the form:

$$(1') \quad \frac{\partial W}{\partial s} = -\frac{1}{\rho W} \frac{\partial P}{\partial s} - \frac{1}{W} \frac{\partial W}{\partial t}.$$

By inserting this value for $\frac{\partial W}{\partial s}$ in equation (10), it becomes:

$$(11) \quad \begin{aligned} & -\frac{1}{\rho W} \left(\frac{a^2 - W^2}{\gamma - 1} \right) \frac{\partial P}{\partial s} \\ & = \left[\frac{1}{W} \frac{a^2 - W^2}{\gamma - 1} + \frac{W}{\gamma - 1} \right] \frac{\partial W}{\partial t} + \frac{dQ - dE}{dt} - \frac{a^2}{\gamma - 1} \frac{1}{M} \frac{dM}{dt} - \frac{a^2 W}{\gamma - 1} \frac{1}{\Sigma} \frac{\partial \Sigma}{\partial s}, \\ & \boxed{-(a^2 - W^2) \frac{\partial P}{\partial s} = \rho a^2 \frac{\partial W}{\partial t} + \rho W \left[(\gamma - 1) \frac{dQ - dE}{dt} - \frac{a^2}{M} \frac{dM}{dt} - a^2 W \frac{1}{\Sigma} \frac{\partial \Sigma}{\partial s} \right]}, \end{aligned}$$

or else, by replacing $W \frac{\Sigma}{s}$ by $\frac{d\Sigma}{dt}$:

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$$(a^2 - W^2) \frac{\partial P}{\partial s} = -\rho a^2 \frac{\partial W}{\partial t} - \rho W \left[(\gamma - 1) \frac{dQ - dE}{dt} - a^2 \left(\frac{1}{M} \frac{dM}{dt} + \frac{1}{\Sigma} \frac{d\Sigma}{dt} \right) \right].$$

2.2.8.1 Conditions Required for Pressure Gradient to Remain Finite

As would be expected, we readily verify that the second term of equation (11) is cancelled for $W=a$ under the same conditions as the second term of equation (10); in other words, the condition required such that the pressure gradient does not become infinite in the cross-section where the fluid velocity becomes equal to the speed of sound is exactly the same as that required so that the velocity gradient does not become infinite in this same cross-section; and that this condition is defined by writing that the second term of equation (10) is equal to zero.

In fact, the two conditions in question are written respectively:

$$\frac{\partial W}{\partial t} = -\frac{1}{W} \left[(\gamma - 1) \frac{dQ - dE}{dt} - a^2 \left(\frac{1}{M} \frac{dM}{dt} + \frac{1}{\Sigma} \frac{d\Sigma}{ds} \right) \right],$$

and

$$\frac{\partial W}{\partial t} = -\frac{a^2}{W} \left[(\gamma - 1) \frac{dQ - dE}{dt} - a^2 \left(\frac{1}{M} \frac{dM}{dt} + \frac{1}{\Sigma} \frac{d\Sigma}{ds} \right) \right],$$

thus for $W=a$:

$$\frac{\partial W}{\partial t} = -\frac{1}{a} \left[(\gamma - 1) \frac{dQ - dE}{dt} - a^2 \left(\frac{1}{M} \frac{dM}{dt} + \frac{1}{\Sigma} \frac{d\Sigma}{ds} \right) \right].$$

2.2.8.2 Pressure Gradient Corresponding to Zero or Negligible Velocity

With all of the terms of the second term of equation (11) containing W as a factor, except for the term $\frac{\partial W}{\partial t}$, we immediately see that the

pressure gradient when flow begins, ie. for zero or negligible velocity, is practically independent of the heat flux provided to the gas, of the chemical reaction which may occur, and even of the convergence or divergence of the tube, such that it depends only on the flow acceleration $\frac{\partial W}{\partial t}$ occurring at the instant considered. In particular,

the pressure gradient is always zero or negligible when flow begins (ie. for zero or negligible velocity) if we have steady state conditions.

2.2.8.3 Other Factors Influencing Pressure Gradient

Finally, we see in equation (11) that the pressure gradient increases (or decreases) with the energy flux transferred to the fluid, and

proportionally is greater when the velocity is closer to a , and that it increases (or decreases) with the convergence or divergence of the tube, depending on whether we find ourselves in the subsonic region (or in the supersonic region).

2.2.8.4 Sign of Pressure Gradient

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As a general rule, the pressure gradient is of opposite sign to that of the velocity gradient existing in the same cross-section, as expected and as proven through comparison of equations (10) and (11).

2.3 INTEGRATION OF MOTION UNDER STEADY STATE CONDITIONS

Neither equation (10) nor equation (11) are susceptible to direct integration, since they contain the "speed of sound" a which is a function of the condition of the fluid, and in particular of its temperature T .

To completely resolve the problem posed, it thus is necessary to discover and combine with equation (10) an independent equation containing only W and T .

It is obvious that the heating of the fluid could be deduced from conservation of energy, and in particular from the application that we can use, by considering the unit mass of the fluid as receiving both heat energy, possibly supplied from the outside, and work, sometimes negative, corresponding to its expansion, and as possessing only as potential energy its substantial internal energy U and its chemical potential energy E , which yields:

$$d\mathcal{G} + dQ = dU + dE$$

or:

$$c dT + p d\mathcal{V} = dQ - dE$$

and, by considering equation (6) and the fact that $\mathcal{V} = \frac{1}{\rho}$

$$\frac{dv}{v} = -\frac{d\rho}{\rho} \quad \text{et} \quad p dv = -\frac{p d\rho}{\rho}$$

$$c \frac{dT}{dt} - \frac{p}{\rho^2} \frac{d\rho}{dt} = \frac{dQ - dE}{dt},$$

or:

$$c \frac{dT}{dt} = \frac{dQ - dE}{dt} + \frac{R}{M} T \frac{1}{\rho} \frac{d\rho}{dt},$$

which, by considering equation (2), finally yields:

$$(12) \quad \boxed{\frac{dT}{dt} = \frac{\partial T}{\partial t} + W \frac{\partial T}{\partial s} = \frac{1}{c} \frac{dQ - dE}{dt} - \frac{R}{cM} T \left(\frac{\partial W}{\partial s} + \frac{W}{\Sigma} \frac{\partial \Sigma}{\partial s} \right)}$$

However, we may transform equation (10) by expressing a as a function of T and by multiplying the two terms by $\gamma - 1$, which yields:

$$(10') \quad \left(\gamma \frac{R}{M} T - W^2 \right) \frac{\partial W}{\partial s} = W \frac{\partial W}{\partial t} + (\gamma - 1) \frac{dQ - dE}{dt} - \gamma \frac{R}{M} T \left(\frac{1}{M} \frac{dM}{dt} + \frac{W}{\Sigma} \frac{\partial \Sigma}{\partial s} \right).$$

In principle, equations (12) and (10') contain only two unknowns W and T which define the mathematical solution to the problem posed. However, it is obvious that solving this system of equations can be done in completely different ways, according to the laws which we assume to define the heat flux $\frac{dQ}{dt}$, the energy released by the chemical reaction

$\frac{dE}{dt}$, the variation of mean molecular mass resulting from this last $\frac{dM}{dt}$, the shape of the tube and the variations over time of the boundary conditions which finally determine the flow acceleration $\frac{\partial W}{\partial t}$. Thus,

it is logical to move directly to the study of the principal interesting special cases.

3. STUDY OF STEADY STATE FLOW

In the case of steady state conditions, the preceding equations obviously are simplified by the fact that the partial differentials for the various variables (W , P , T) become zero with respect to time, and an additional simplification of the problem obviously results from the fact that the mass-flow μ is the same throughout all of the cross-sections of the tube.

This last observation allows us to perform the calculation in a different manner, with greater advantage than by specifying the general equations (10) or (10'), (11) and (12) above, and we also will see that it becomes possible to extend the solution much further without great difficulty. We will accomplish this through successive application of the differential equations and through application of the full equations.

3.1 DIFFERENTIAL EQUATIONS CORRESPONDING TO MOST GENERAL CASE OF STEADY STATE CONDITIONS

3.1.1 General Equation for Fluid Mechanics

This time this is written:

$$(1) \quad W dW = -\frac{dP}{\rho}.$$

3.1.2 Continuity Equation

This time this is written:

$$(2) \quad \mu = \Sigma \rho W,$$

or:

$$(2') \quad \frac{d\Sigma}{\Sigma} + \frac{d\rho}{\rho} + \frac{dW}{W} = 0.$$

3.1.3 Conservation of Energy Equation

$$(3) \quad W dW = dQ - dH - dE = dQ - dE - C dT$$

3.1.4 State Equation

This remains unchanged and is still written:

$$(4) \quad \frac{P}{\rho} = \frac{R}{M} T \quad \text{or} \quad \frac{dP}{P} = \frac{d\rho}{\rho} - \frac{dM}{M} + \frac{dT}{T}.$$

3.1.5 Differential Equation Defining Temperature Variation

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From equation (2), we deduce:

$$\frac{1}{\rho} = \frac{\Sigma}{\mu} W,$$

and from (2'):

$$-dW = \left(\frac{d\rho}{\rho} + \frac{d\Sigma}{\Sigma} \right) W.$$

By considering (1), it becomes:

$$(5) \quad W^2 \left(\frac{d\rho}{\rho} + \frac{d\Sigma}{\Sigma} \right) = \frac{dP}{\rho} = + \frac{\Sigma}{\mu} W dP$$

or, by considering (2):

$$\left(\frac{d\rho}{\rho} + \frac{d\Sigma}{\Sigma} \right) = \frac{\Sigma^2}{\mu^2} \rho dP,$$

or

$$\frac{d\rho}{\rho} + \frac{d\Sigma}{\Sigma} = \frac{\Sigma^2}{\mu^2} P \rho \frac{dP}{P}$$

and by considering (4'):

$$\frac{dP}{P} + \frac{dM}{M} - \frac{dT}{T} + \frac{d\Sigma}{\Sigma} = \frac{\Sigma^2}{\mu^2} P \rho \frac{dP}{P},$$

or:

$$(6) \quad \left(\frac{\Sigma^2}{\mu^2} P \rho - 1 \right) \frac{dP}{P} = \frac{dM}{M} - \frac{dT}{T} + \frac{d\Sigma}{\Sigma}.$$

However, on the other hand, we deduce from (1) and (3):

$$\frac{P}{\rho} \frac{dP}{P} = C dT - (dQ - dE),$$

where:

$$(7) \quad \frac{dP}{P} = \frac{M}{RT} [C dT - (dQ - dE)].$$

By replacing $\frac{dP}{P}$ from (7) in equation (6), it becomes:

$$(8) \quad \left(1 - \frac{\Sigma^2}{\mu^2} P \rho \right) \frac{M}{RT} [(dQ - dE) - C dT] = \frac{dM}{M} - \frac{dT}{T} + \frac{d\Sigma}{\Sigma},$$

or:

$$(8') \quad \left(\frac{M}{RT} - \frac{\Sigma^2}{\mu^2} \rho^2 \right) [(dQ - dE) - CT \frac{dT}{T}] = \frac{dM}{M} - \frac{dT}{T} + \frac{d\Sigma}{\Sigma}.$$

However:

$$\gamma \frac{P}{\rho} = a^2 = \gamma \frac{R}{M} T,$$

and:

$$\frac{\Sigma \rho}{\mu} = \frac{1}{W},$$

such that (8') becomes:

$$\frac{dT}{T} \left[1 - CT \left(\frac{\gamma}{a^2} - \frac{1}{W^2} \right) \right] = - \left(\frac{\gamma}{a^2} - \frac{1}{W^2} \right) (dQ - dE) + \frac{dM}{M} + \frac{d\Sigma}{\Sigma}$$

or, by considering that:

$$CT = C \frac{M P}{R \rho} = \frac{a^2}{\gamma - 1}$$

and by simplifying:

$$(9) \quad \boxed{\frac{1}{\gamma - 1} \left[\frac{a^2}{W^2} - 1 \right] dT = \frac{M}{R} \left(\frac{a^2}{\gamma W^2} - 1 \right) (dQ - dE) + T \left(\frac{dM}{M} + \frac{d\Sigma}{\Sigma} \right)}$$

We already see in equation (9) that the coefficient of $dQ - dE$ is cancelled for $W^2 = \frac{a^2}{\gamma}$, while the coefficient of dT is not cancelled,

which could mean that the energy flux will have no influence on the temperature variation at the moment where the velocity W reaches the value a , while the variation of molecular mass and the convergence

or divergence of the tube will continue to influence the temperature gradient for this velocity value.

We also see in equation (9) that the coefficient of dT is cancelled for $W=a$, while those for the second term are not cancelled, which corresponds to the condition already mentioned in the first part of the present study, with respect to the passage through the "speed of sound", and according to which a precise relationship must exist between the energy flux received by the fluid, the variation of mean molecular mass, and the convergence or divergence of the tube, such that the velocity and pressure gradients, and thus temperature gradients, do not become infinite in the absence of flow acceleration, ie. under steady state conditions.

In a more precise fashion, the condition in question obviously is expressed under steady state conditions by the fact that the second term of the equation must be zero for $W=a$, ie. through the relationship:

$$(10) \quad \frac{M}{R} \frac{\gamma-1}{\gamma} (dQ - dE) - T \left(\frac{dM}{M} + \frac{d\Sigma}{\Sigma} \right) = 0,$$

or:

$$(10') \quad (dQ - dE) + \frac{a^2}{\gamma-1} \left(\frac{dM}{M} + \frac{d\Sigma}{\Sigma} \right) = 0.$$

This condition is identified, as can be readily verified, with that which we have obtained by setting the second terms of equations (10) or (11) of paragraph 2.2 to zero and by having $\frac{W}{t}=0$ and $W=a$. /20

We also see in equation (9) that the direction of the temperature variation changes, all other things being equal (ie. for the same sign of the second term), when W passes through the value a corresponding to the "speed of sound". On the contrary, dT retains its sign when W reaches the value a if the second term changes sign for $W=a$ (ie. if the velocity gradient itself remains of constant sign throughout the entire length of the tube).

However, equation (9) may be transformed by multiplying the two terms by $(\gamma-1) W^2$, and by observing that:

$$(\gamma-1) \frac{M}{R} = \frac{C-c}{c} \frac{1}{C-c} = \frac{1}{c},$$

which yields:

$$(9') \quad (a^2 - W^2) dT = (a^2 - \gamma W^2) \left(\frac{dQ - dE}{C} \right) + (\gamma-1) T W^2 \left(\frac{dM}{M} + \frac{d\Sigma}{\Sigma} \right).$$

We discover in equation (9') the fact that heat addition or a chemical reaction without any variation of mean molecular mass have no effect on the temperature of the fluid for $W = \frac{a}{\sqrt{\gamma}}$ and that the temperature gradient

becomes infinite for $W=a$ if condition (10) or (10') is not fulfilled.

In addition, we see in (9') that for very small velocity values, ie. when flow begins, the convergence or divergence of the tube and the variation of mean molecular mass have practically no influence on the variations of the fluid temperature, which at this moment are the same as with heating at constant pressure, since for $W=0$, we have:

$$dT = \frac{dQ - dE}{C}$$

Later, we will effectively discover the fact that the flow always begins to be isobaric for $W=0$, except in the very special case of isentropic expansion.

3.1.6 Differential Equation Defining Velocity Variations

Equation (3) obviously may be written:

$$\frac{dW}{W} = \frac{1}{W^2} W dW = \frac{1}{W^2} (dQ - dE - C dT)$$

or, by taking $C dT$ from equation (9'):

$$\frac{dW}{W} = \frac{dQ - dE}{W^2} \left[1 - \gamma \frac{\frac{a^2}{\gamma} - W^2}{a^2 - W^2} \right] - \frac{C(\gamma - 1)T}{a^2 - W^2} \left(\frac{dM}{M} + \frac{d\Sigma}{\Sigma} \right)$$

or, by observing that:

$$C(\gamma - 1)T = \gamma(C - c)T = \gamma \frac{R}{M} T = a^2,$$

$$\frac{dW}{W} = \frac{\gamma - 1}{a^2 - W^2} (dQ - dE) - \frac{a^2}{a^2 - W^2} \left(\frac{dM}{M} + \frac{d\Sigma}{\Sigma} \right),$$

or:

$$(11) \quad \frac{dW}{W} = \frac{a^2}{a^2 - W^2} \left[\frac{dQ - dE}{CT} - \frac{dM}{M} - \frac{d\Sigma}{\Sigma} \right].$$

This equation could have been obtained, as can be readily verified by setting $\frac{\partial W}{\partial t} = 0$ and by observing that under steady state conditions

$\frac{\partial W}{\partial s} = \frac{dW}{ds} = \frac{dW}{dt} \frac{dt}{ds} = \frac{1}{W} \frac{dW}{dt}$, directly from equation (10) transformed from paragraph 2.2.6, corresponding to the general case of the variable state

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condition.

We discover in equation (11) that the velocity gradient becomes infinite for $W=a$, if however the parenthetical term is not also cancelled for $W=a$.

We also discover that the velocity gradient changes sign for $W=a$, if the parenthetical portion of the second term does not change sign under the same conditions.

In particular, we see that in the absence of energy addition and variation of mean molecular mass, $\frac{d\Sigma}{\Sigma}$ must pass from negative values to positive values by being cancelled for $W=a$ (ie. that there must be a "throat" for the passage through the "speed of sound", such that the velocity gradient is constantly positive throughout the tube).

In addition, we see that if there is energy addition (in the form of heat addition or chemical reaction) and if we neglect the possible variation of molecular mass resulting from this last, the condition such that the velocity gradient does not become infinite for $W=a$ is that the relative variation of cross-section of the tube $\frac{d\Sigma}{\Sigma}$, corresponding to

the cross-section where the fluid velocity is equal to the "speed of sound" is equal to the relative variation of absolute temperature $\frac{dT}{T}$ of the fluid which would result from the energy flux which is provided to it (directly or through the chemical reaction which is produced), with heating at constant pressure.

In the special case considered for steady state flow, the equation demonstrates in a more precise fashion that energy addition (having the consequence of rendering the sum of the terms in dQ , dE and dM different from zero) has the effect of moving the cross-section where the fluid attains the "speed of sound" to beyond the throat (if the energy addition is positive) and short of this (if the energy addition is negative), such that the throat itself no longer corresponds to the "speed of sound".

Finally, and in a general fashion, we discover the fact that the velocity varies in the same way as in an isentropic flow where the divergence of the tube would have the value:

$$\frac{d\Sigma}{\Sigma} + \frac{dM}{M} - \frac{dQ - dE}{CT}$$

3.1.7 Differential Equation Defining Pressure Variations

Equation (1) may be written:

$$W dW = -\frac{P}{\rho} \frac{dP}{P} = -\frac{a^2}{\gamma} \frac{dP}{P}$$

Thus, we have:

$$(12) \quad \frac{dP}{P} = \frac{\gamma W^2}{a^2 - W^2} \left[\frac{d\Sigma}{\Sigma} + \frac{dM}{M} - \frac{dQ - dE}{CT} \right].$$

As above, we observe that equation (12) could be deduced directly from equation (11) transformed from paragraph 2.2.10, corresponding to the variable state condition, by having $\frac{\partial W}{\partial t} = 0$ and $\frac{\partial W}{\partial s} = \frac{1}{W} \frac{dW}{dt}$.

It is useless to persist further with the discussion of this equation, since in any case, equation (1) demonstrates that dp and dW are constantly of opposite signs. This point will be refined later.

However, we cannot fail to observe that according to equation (12) $\frac{dp}{p}$ must be zero for $W=0$, thus negligible when fluid flow begins.

In addition, we verify in equation (12) that the relative pressure variation tends to become infinite and to change sign for $W=a$, remaining finite at this moment only if the parenthetical portion of the second term is cancelled for $W=a$ and retains its sign only if in addition said parenthetical term changes sign upon passage through the speed of sound.

As would be expected, the conditions required so that the pressure gradient remains finite or remains with the same sign throughout the entire length of the tube, thus are exactly the same as those which are required so that the velocity gradient of the fluid remains finite, or of constant sign, also throughout the entire length of the tube.

3.1.8 Differential Equation Defining Fluid Density Variations

It suffices to note that according to equation (2'):

$$\frac{d\rho}{\rho} = -\left(\frac{dW}{W} + \frac{d\Sigma}{\Sigma}\right),$$

from which we deduce:

$$(13) \quad \frac{d\rho}{\rho} = \frac{a^2}{a^2 - W^2} \left[\frac{d\Sigma}{\Sigma} + \frac{dM}{M} - \frac{dQ - dE}{CT} \right] - \frac{d\Sigma}{\Sigma}.$$

Direct discussion of equation (13) would not offer much additional benefit, given that which has already been provided for equations (11) and (12). On the contrary, the expression $\frac{d\rho}{\rho}$ will permit us to calculate the polytropic coefficient n for a flow "tangent" to that of the fluid in the cross-section considered.

Finally, we observe that for pressure variations, as for velocity variations, everything occurs as for the isentropic expansion of a

constant composition fluid, except that the divergence of the tube must be replaced by the expression:

However, of course this analogy pertains to the velocity and pressure variations corresponding to a cross-section Σ at actual velocities W and sonic velocities a and at a pressure P .

3.1.9 Value of Polytropic Exponent "Tangent" to Fluid Flow in Cross-Section Considered

It is obvious that we always may represent the fluid flow by a polytropic equation $p\varrho^n = C^{te}$ in an infinitely small interval in the vicinity of the cross-section considered, on the condition that we select an appropriate value for n , ie. defined by the obvious relationship:

$$n = \frac{\frac{dP}{P}}{\frac{d\varrho}{\varrho}}$$

Under these conditions, we also deduce from (12) and (13):

$$(14) \quad n = \frac{\gamma}{a^2} W^2 \frac{-\frac{dM}{M} + \frac{dQ-dE}{CT} - \frac{d\Sigma}{\Sigma}}{-\frac{dM}{M} + \frac{dQ-dE}{CT} - \frac{W^2 d\Sigma}{a^2 \Sigma}}$$

Equation (14) clearly provides very interesting indications on the flow of the fluid through the tube.

In fact, we see the following in this equation.

3.1.9.1 Value Corresponding to $W=0$

-that for $W=0$, and if $\frac{dM}{M} - \frac{dQ-dE}{CT}$ is not precisely zero, n is always zero; ie. that when motion begins, and when the velocity of the fluid remains low with respect to the speed of sound, its flow is practically isobaric, regardless of the shape of the corresponding portion of the tube and the conditions under which the flow occurs (provided that it is not isentropic within the limited sense of the word).

3.1.9.2 Values Corresponding to Interval $W=0, W=a$

-that when W varies between 0 and a , we see in formula (14) that the

variations of n result from those of $\frac{W^2}{a^2}$ on the one hand, and from those of a fraction which we may consider as resulting from the addition of the same quantity $\left(\frac{dQ-dE}{CT} - \frac{dM}{M}\right)$ to the numerator and the denominator of the fraction:

$$\frac{-\frac{d\Sigma}{\Sigma}}{-\frac{W^2}{a^2} \frac{d\Sigma}{\Sigma}} = \frac{a^2}{W^2}.$$

From this, it turns out that if the velocity increases constantly in this portion of the flow (which according to (11) requires that the numerator of said fraction is always positive and consequently also the denominator if we assume $\frac{dQ-dE}{CT} - \frac{dM}{M} \geq 0$), n remains of necessity between the two boundary values: $\gamma \frac{W^2}{a^2} \leq n \leq \gamma$ if $\frac{d\Sigma}{\Sigma} \leq 0$; these values are in fact precisely realized:

-the first, when $\frac{d\Sigma}{\Sigma}$ is constantly zero (special case of the cylindrical tube;

-the second, when $\frac{dQ-dE}{CT} - \frac{dM}{M}$ is constantly zero (special case of the isentropic expansion of a constant composition fluid).

In fact, we see in addition, according to formula (14), that if the quantities $\left(\frac{dQ-dE}{CT} - \frac{dM}{M}\right)$ and $\frac{d\Sigma}{\Sigma}$ do not vary too rapidly, and if the preceding hypotheses are always realized, the polytropic coefficient n , characterizing the flow of the fluid, has a tendency to increase constantly with the Mach number $\frac{W}{a}$, starting from $n=0$ for $\frac{W}{a}=0$ (except naturally for the special case of the isentropic expansion of a constant composition fluid).

In a more precise manner, we may observe that if $\left(\frac{dQ-dE}{CT} - \frac{dM}{M}\right)$ and $\frac{d\Sigma}{\Sigma}$ are constant within a certain interval, the "energy addition" being constantly positive, and the velocity constantly increasing, n increases constantly with $\frac{W^2}{a^2}$ (and even more quickly than $\frac{W^2}{a^2}$ if the divergence of the tube $\frac{d\Sigma}{\Sigma}$ is positive, the fraction which appears in expression (14) in this case being an increasing function of $\frac{W}{a}$).

If the heat addition or chemical reaction which may affect the flow and if the divergence of the tube vary in an absolutely random manner, we may simply confirm that n deviates from zero for $W=0$ if the flow is not adiabatic, and that for $W < a$, n is less than or equal to $\gamma \frac{W^2}{a^2}$ if

$\left(\frac{dQ-dE}{CT} - \frac{dM}{M}\right)$ and $\frac{d\Sigma}{\Sigma}$ have the same sign, and are positive, and are comprised between $\gamma \frac{W^2}{a^2}$ and γ , if the same quantities have opposite signs (because n may assume any finite value, positive or negative, if the two quantities in question are negative simultaneously).

3.1.9.3 Values Corresponding to $W=a$

-that for $W=a$, n occurs, according to formula (14) and in the general case, as an indeterminate form if the flow is steady state, because the numerator of the fraction appearing in expression (14) must become zero for $W=a$ (such that $\frac{dW}{W}$ can remain finite and also consequently $\frac{dP}{P}$, in the absence of any acceleration of the flow) and consequently, the denominator of this same fraction.

Again in the general case of steady state flow, we also may observe that by virtue of what occurs, the relative variation of cross-section in the region of the tube where the fluid attains the "speed of sound" $W=a$ is defined by the condition:

or, if we prefer:

$$\frac{d\Sigma}{\Sigma} = \frac{dQ - dE}{CT} - \frac{dM}{M}$$

$$\frac{1}{\Sigma} \frac{d\Sigma}{ds} = \frac{1}{CT} \left(\frac{dQ}{ds} - \frac{dE}{ds} \right) - \frac{1}{M} \frac{dM}{ds};$$

in such a manner that we may set $\frac{W^2}{a^2} = 1 + \epsilon$, ϵ approaching zero when W approaches a , and $\frac{dQ-dE}{CT} - \frac{dM}{M} \frac{d\Sigma}{\Sigma} = \epsilon'$, ϵ' approaching zero when W approaches a .

With these notations, we obviously have:

$$n = \gamma (1 + \epsilon) \frac{\epsilon'}{\epsilon' + \left(1 - \frac{W^2}{a^2}\right) \frac{d\Sigma}{\Sigma}} = \gamma (1 + \epsilon) \frac{\epsilon'}{\epsilon' - \epsilon \frac{d\Sigma}{\Sigma}}$$

or finally:

(15)

$$n = \gamma \frac{1 + \epsilon}{1 - \frac{\epsilon}{\epsilon'} \frac{d\Sigma}{\Sigma}}$$

Formula (15) defines the value of n perfectly, as we have seen, $\frac{d\Sigma}{\Sigma}$ having a perfectly defined value for $W=a$:

$$\frac{d\Sigma}{\Sigma} = \frac{dQ - dE}{CT} - \frac{dM}{M},$$

however on the condition that we are given in addition the boundary of the ratio of the infinitely small values and ' when W approaches a

(and this confirms how and why the value for the polytropic coefficient n also depends on an indeterminate form for $W=a$).

In a more precise manner, we may calculate without ambiguity the value of the exponent n for the polytropic tangent in the cross-section $W=a$ when we define special conditions for the flow:

-if in fact we assume that the "energy addition" $\left(\frac{dQ-dE}{CT} - \frac{dM}{M}\right)$ is constantly zero during the flow (special case for isentropic flow of a constant composition gas), we have:

$$n = \gamma \frac{W^2}{a^2} \frac{-\frac{d\Sigma}{\Sigma}}{-\frac{W^2}{a^2} \frac{d\Sigma}{\Sigma}} = \gamma$$

If we assume that the cross-section Σ is constant (cylindrical tube), formula (14) also is simplified and rigorously yields:

$$n = \gamma \frac{W^2}{a^2},$$

thus:

$$n = \gamma \text{ for } W=a.$$

We will find these results acceptable for the corresponding special cases, but it is possible to define more precisely what occurs in the general case at the moment where the velocity attains the critical value a .

In fact, equation (11) may be written:

$$\frac{dW}{W} = \frac{1}{1 - \frac{W^2}{a^2}} \left[\frac{dQ - dE}{CT} - \frac{dM}{M} - \frac{d\Sigma}{\Sigma} \right]$$

or, for $W=a$:

$$\frac{dW}{W} = -\frac{\epsilon'}{\epsilon},$$

where:

$$n = \frac{1}{1 + \frac{W}{dW} \frac{d\Sigma}{\Sigma}} \quad (W=a)$$

and equation (12):

$$\frac{dP}{P} = -\gamma \frac{W^2}{a^2} \frac{dW}{W} = +\gamma \frac{\epsilon'}{\epsilon} \quad (\text{for } W=a).$$

Finally, $\frac{d\rho}{\rho}$ is deduced from equation (2') by writing:

$$\frac{d\rho}{\rho} = -\frac{dW}{W} - \frac{d\Sigma}{\Sigma} = +\frac{\varepsilon'}{\varepsilon} - \frac{d\Sigma}{\Sigma} \quad (\text{for } W=a).$$

3.1.9.4 Values Corresponding to $W>a$

-that for $W>a$, and naturally to the extent where this condition may be present without the cessation of steady state conditions, the quantities $\frac{dM}{M} \frac{dQ-dE}{CT}$ and $\frac{d\Sigma}{\Sigma}$ assumed to be different from zero, we see directly in equation (14) that n always will be greater than or equal to $\gamma \frac{W^2}{a^2}$ if $\frac{dQ-dE}{CT} \frac{dM}{M}$ and $\frac{d\Sigma}{\Sigma}$ are positive (however, on the condition that the numerator and denominator of the fraction appearing in equation (14) are both positive) and that n will fall between γ and $\gamma \frac{W^2}{a^2}$ in the opposite case¹.

3.2 DIFFERENTIAL EQUATIONS YIELDING VELOCITY AND TEMPERATURE

The differential equations above have been established solely in view of permitting a simple discussion of the development of the local temperature, velocity, pressure, density or polytropic coefficient, but in reality they are not suitable for the calculation of the values attained in each cross-section for these different flow characteristics (except possibly for the last one), due to the fact that the square of the "speed of sound" a^2 , which appears in all of these equations, is a dependent variable for both upstream conditions and for the flow of the fluid between the inlet to the tube and the cross-section considered.

¹Note: These conclusions clearly appear if we note that according to formula (14), the polytropic coefficient n may be considered as formed from the product of $\gamma \frac{W^2}{a^2}$ by a fraction:

$$\frac{dQ-dE}{CT} \frac{dM}{M}$$

(which may be positive or negative), such that the fraction obtained approaches or deviates more or less from unity and that its product by $\gamma \frac{W^2}{a^2}$ becomes greater than or less than γ .

Thus, it is necessary to proceed directly with a different elimination to obtain equations capable of actually being integrated, in view of furnishing the value for the velocity, pressure or temperature in a given cross-section.

3.2.1 Differential Equation Yielding Velocity

An initial method consists of starting with the following four fundamental equations, in which the quantities W_0 , E_0 , T_0 , P_0 , ρ_0 , correspond to the velocity, chemical potential energy, temperature, pressure and density, when the flow begins, ie. in the cross-section of the inlet to the tube Σ_0 , and where Q represents the total quantity of heat transferred per unit of mass of the fluid between this inlet cross-section Σ_0 and the cross-section Σ considered:

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$$(3') \quad \frac{W^2 - W_0^2}{2} = Q + E_0 - E + H_0 - H,$$

$$W dW = dQ - dE - dH;$$

$$(1) \quad W dW = -\frac{dP}{\rho}.$$

$$\frac{W^2 - W_0^2}{2} = -\int \frac{dP}{\rho};$$

$$(2) \quad \frac{d\Sigma}{\Sigma} + \frac{d\rho}{\rho} + \frac{dW}{W} = 0, \quad \mu = \rho \Sigma W;$$

$$(4) \quad \frac{P}{\rho} = \frac{R}{M} T, \quad \frac{dP}{P} = \frac{d\rho}{\rho} - \frac{dM}{M} + \frac{dT}{T}.$$

By considering (2), we note that equation (1) may be written:

$$(16) \quad dP = -\frac{\mu}{\Sigma} dW,$$

which is integrated to yield:

$$(17) \quad P_0 = P + \mu \int_{\Sigma}^1 \frac{1}{\Sigma} dW.$$

By considering (4) and (17), we then may express the temperature T in such a manner so as to define the enthalpy H in equation (3'), which yields:

$$H = CT = C \frac{M}{R} \frac{P_0 - \mu \int_{\Sigma}^1 \frac{1}{\Sigma} dW}{\frac{\mu}{\Sigma W}}.$$

Replacing this value of H in equation (3'), we have:

$$\frac{W^2 - W_0^2}{2} = Q + E_0 - E + H_0 - C \frac{M \Sigma W}{R \mu} \left[P_0 - \mu \int_{\Sigma} \frac{1}{\Sigma} dW \right],$$

or:

$$(18) \quad \frac{W^2 - W_0^2}{2} = Q + E_0 - E + H_0 - \frac{\gamma}{\gamma - 1} \frac{\Sigma W}{\mu} \left[P_0 - \mu \int_{\Sigma} \frac{1}{\Sigma} dW \right].$$

Noting that the initial potential energy of the fluid has a value:

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$$\epsilon_0 = H_0 + \frac{W_0^2}{2}$$

(and this regardless of its initial physical state) and that the energy added to the fluid since its entrance into the tube up to the cross-section Σ considered has a value:

$$\epsilon = Q + E_0 - E,$$

finally, that W obviously is a function of the abscissa s of said cross-section Σ , we may write (18) in the form:

$$(19) \quad \boxed{\frac{W^2}{2} + \frac{\gamma}{\gamma - 1} \frac{\Sigma}{\mu} P_0 W - \frac{\gamma}{\gamma - 1} \Sigma W \int_{\Sigma} \frac{1}{\Sigma} \frac{\partial W}{\partial s} ds = \epsilon_0 - \epsilon}$$

Equation (19) obviously constitutes the differential equation to be integrated to obtain the values for W in the different cross-sections Σ corresponding to the different abscissa s of the tube, provided that we assume that the energy ϵ added to the fluid is determined as a function of the abscissa s considered.

We will see later that this differential equation can be integrated effectively in an immediate fashion in the special case where the cross-section Σ is constant, and then is reduced to an ordinary second degree equation with W.

3.2.2 Differential Equation Yielding Temperature

By considering equation (3'), equation (11) becomes, by replacing a^2 by $\frac{\gamma R T}{M}$ and with the same notations as above:

$$(20) \quad W dW = W^2 \frac{dW}{W} = \frac{2(\epsilon_0 + \epsilon - CT)}{1 - \frac{\gamma R T}{M}} \left[\frac{d\Sigma}{\Sigma} + \frac{dM}{M} - \frac{d\epsilon}{CT} \right] = d\epsilon - C dT$$

or, with all simplifications performed:

$$(21) \quad \left[\gamma \frac{R}{M} T - 2(\xi_0 + \xi - CT) \right] C dT + d\xi \left[\gamma \frac{R}{M} T - 2(\gamma - 2)(\xi_0 + \xi) \right] \\ + 2\gamma \frac{R}{M} T (\xi_0 + \xi - CT) \cdot \left[\frac{1}{\Sigma} \frac{d\Sigma}{ds} ds + \frac{1}{M} \frac{dM}{ds} ds \right] = 0.$$

The values for Σ , ξ and M assumed to be known as a function of the abscissa s , equation (21) will constitute the differential equation to be solved to find the temperature T as a function of this same abscissa.

3.3 CONCLUSIONS CONCERNING GENERAL CASE OF STEADY STATE FLOW

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If we consider the general case of steady state flow in any type of tube, ie. if we exclude the special cases (which will be studied later) where the quantities $\left(\frac{dM}{M} - \frac{dQ}{CT} - \frac{dE}{\Sigma} \right)$ or $\frac{d\Sigma}{\Sigma}$ are zero, or if we prefer, if we restrict ourselves to the truly general case where the tube has any shape and where there is energy addition (or energy loss) in the form of heat flux, or through chemical reaction, we may summarize or add to the discussion of the different formulae or equations above by stating the following conclusions.

3.3.1 Free Variation of Parameters for $W=a$

When the velocity of the fluid is different from the speed of sound (ie., in the regions of the tube where W is different from a), the heat flux transferred to the fluid, the intensity of the chemical reaction capable of occurring within the fluid, the variation of mean molecular mass resulting from said chemical reaction, and the convergence or divergence of the tube, may assume any values without the risk of any discontinuity in the expansion of the fluid through the tube, even in the case considered where we require that steady state conditions prevail, with the sole conditions that the flow must remain laminar and that the convergence or divergence of the tube $\frac{d\Sigma}{\Sigma}$ remains finite.

3.3.2 Conditions Required for $W=a$

The velocity of the fluid cannot pass the value $W=a$, corresponding to the speed of sound, without the fluid ceasing to remain under steady state conditions, unless the quantity:

$$(22) \quad \left(\frac{dQ}{CT} - \frac{dE}{M} - \frac{d\Sigma}{\Sigma} \right)$$

is precisely cancelled for $W=a$.

3.3.3 Sign of Velocity Gradient

For a given sign for the expression in question, the velocity gradient has the opposite sign as long as the flow velocity is less than or greater than the speed of sound. In particular, if expression (22) is positive, the velocity gradient is positive with subsonic flow and negative with supersonic flow, and the opposite is true if expression (22) is negative. In particular, it turns out that the velocity gradient may change sign, both in the subsonic portion and possibly in the supersonic portion of the flow, if the laws required for the quantities ρ , M and Σ are such that expression (22) changes sign for velocity values other than $W=a$.

3.3.4 Condition with Constantly Increasing Velocity

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The conditions required so that the velocity may increase constantly throughout the tube and pass the speed of sound without ceasing to increase and without ceasing to have steady state conditions are that the quantity (22) is cancelled for $W=a$ and changes sign when W passes this value, such that it is always positive for $W<a$ and always negative for $W>a$; such that everything occurs, from the point of view of the variations of velocity (and pressure) and in each cross-section Σ , as if it was a question of isentropic flow of a constant composition gas in a tube whose divergence, corresponding to the cross-section considered, was defined by expression (22) with the sign changed.

3.3.5 Sign of Pressure Gradient

For steady state flow, the pressure gradient is of opposite direction to the velocity gradient at any instant and any point, regardless of local energy addition and the convergence or divergence of the tube.

3.3.6 Variations of Temperature and Polytropic Coefficient

3.3.6.1 Beginning of Motion

At the beginning of motion, ie. for zero or negligible fluid velocities, the polytropic coefficient corresponding to a fluid flow having the same ratio $\frac{dp}{d\psi}$ as the actual flow generally begins by

being zero (ie. the flow is isobaric, at least when the quantity $\frac{dM}{M} \frac{dQ-dE}{CT}$ is found to be precisely zero at the same moment), and

consequently the fluid temperature begins by increasing if there is positive energy addition in the form of heat through chemical reaction (ie. if $dQ-dE>0$), the heating of the fluid naturally being equal to the energy quotient which is transferred to it through the specific heat at constant pressure C , as equation (9') demonstrates.

3.3.6.2 For

Later, ie. for increasing values of W and $\frac{W}{a}$, the polytropic coefficient, or at least the temperature, generally continues to increase up to the moment where W attains a critical value, which is even lower when the tube is more convergent, and the energy addition (assumed positive) is low. For this critical value of W [which is that which cancels the second term of equation (9')], the fluid temperature passes through a maximum and the polytropic coefficient would pass through the value 1 if the variation of mean molecular mass was negligible [as we readily verify by replacing $\frac{dQ-dE}{CT}$ by the value cancelling the second term of (9') in expression (14)].

In any case, when W attains the value $\frac{a}{\sqrt{\gamma}}$ the temperature gradient no longer depends on the energy addition pertaining to the corresponding cross-section, its value being determined primarily by the convergence or divergence of the tube, and secondarily by the variation of the mean molecular mass, such that its sign depends only on the algebraic sum:

$$\frac{dM}{M} + \frac{d\Sigma}{\Sigma}$$

for this particular velocity value.

3.3.6.3 For

Past the critical value $\frac{1}{\sqrt{\gamma}}$ in question, both $\frac{W}{a}$ and W remain smaller than a , the influence of energy addition on the temperature variations of the fluid become negative, such that the temperature of the fluid definitely decreases if $\frac{d\Sigma}{\Sigma}$ is less than or simply equal to zero (convergent or cylindrical shaped tube) from the moment that the energy addition is not really negative.

It is obvious that with $\frac{W}{a}$ always varying between the critical value $\frac{1}{\sqrt{\gamma}}$ in question and one, the hypotheses above always being realized and $\frac{dM}{M}$ remaining negligible with respect to $\frac{dQ-dE}{CT} \frac{d\Sigma}{\Sigma}$, the polytropic coefficient n deviates from one and remains greater than this, while the pressure decreases, since in each cross-section, it is possible to write:

$$\frac{dP}{P} - n \frac{d\rho}{\rho} = 0,$$

and

$$\frac{dT}{T} = \frac{dP}{P} - \frac{d\rho}{\rho} + \frac{dM}{M} = \left(1 - \frac{1}{n}\right) \frac{dP}{P} + \frac{dM}{M},$$

such that if $\frac{dP}{P}$ is negative and $\frac{dT}{T}$ is negative, n necessarily must be greater than 1 if $\frac{dM}{M}$ remains negligible with respect to the other terms of the last equation.

3.3.6.4 For

In addition, we may note that for $\frac{W}{a}$ slightly less than 1, n necessarily assumes a value close to γ if $\frac{dQ-dE}{CT} \frac{dM}{M} \frac{d\Sigma}{\Sigma}$ is not small, and that it is only when this quantity must be cancelled so that steady state conditions remain, ie. when $\frac{W}{a}$ becomes exactly equal to 1, that expression (14) becomes an indeterminate form whose value has been defined in paragraph 3.1.9.3, formula (15).

3.3.6.5 For

Finally, for $\frac{W}{a}$ greater than 1, the influence of energy addition $\frac{dQ-dE}{CT}$ on the temperature variations of the fluid once again changes direction, as equation (9') demonstrates, such that if this energy addition is positive, it once again contributes to increase the temperature of the fluid, and this effectively increases if the divergence of the tube is not too great [ie., if $\frac{d\Sigma}{\Sigma}$ is less than the critical value in the cross-section considered and for which the second term of equation (9') is cancelled].

Under the same conditions, ie. always for $\frac{W}{a} > 1$, we have already seen (3.1.9.4) that the polytropic coefficient n remains greater than or equal to $\gamma \frac{W^2}{a^2}$ if $\frac{dQ-dE}{CT} \frac{dM}{M}$ and $\frac{d\Sigma}{\Sigma}$ are positive (energy addition in divergent tube or energy subtraction in convergent tube) provided that the numerator and denominator of formula (14) are both positive, while it becomes less than or equal to $\gamma \frac{W^2}{a^2}$ if the two quantities in question are of opposite signs (energy addition in convergent tube or energy subtraction in divergent tube).

In addition, equation (11) demonstrates that for Mach numbers greater than one, ie. for fluid velocities greater than the "speed of sound", the velocity ceases to increase and decrease even if the divergence of the tube becomes sufficiently small with respect to the assumed positive energy addition, the pressure gradient naturally being reversed in this case and the temperature becoming increasing.

3.3.7 Influence of Mass-Flow on Upstream-Downstream Pressure Difference

It is nearly obvious, and easy to verify in these cases, that all other things being equal, the total pressure difference between the upstream cross-section and any other section of the tube increases rapidly with the mass-flow μ and naturally also the velocities W realized in each section.

From equation (1) $W dW = \frac{dP}{\rho}$ and equation (2) $\mu = \rho \Sigma W$, we deduce in fact:

$$(16) \quad dP = - \frac{\mu}{\Sigma} dW,$$

such that if we designate the upstream pressure by P_0 , we have:

$$P = P_0 - \mu \int \frac{1}{\Sigma} dW,$$

and that the pressure difference between the upstream section and any other section of the tube has the value:

$$(24) \quad \boxed{P_0 - P = \mu \int \frac{1}{\Sigma} dW},$$

an equation which yields the value of $P - P_0$ without difficulty, once the velocity variation law is determined, for example through integration of equations (19) or (20) and which already demonstrates unquestionably that $(P - P_0)$ increases, all other things being equal, with the flow μ .

Equation (24) also may be transformed by considering equation (11), in such a manner to demonstrate even more clearly the influence of the flow on the pressure difference required to obtain this flow.

By replacing dW by its value drawn from (11) in equation (24), we then obtain: /34

$$P_0 - P = \mu \int \frac{W}{\Sigma} \frac{a^2}{a^2 - W^2} \left[\frac{dQ - dE}{CT} - \frac{dM}{M} - \frac{d\Sigma}{\Sigma} \right]$$

or by considering (2):

$$P_0 - P = \mu^2 \int \frac{1}{\rho \Sigma^2} \frac{a^2}{a^2 - W^2} \left[\frac{dQ - dE}{CT} - \frac{dM}{M} - \frac{d\Sigma}{\Sigma} \right].$$

3.4 PRINCIPAL SPECIAL CASES PERTAINING TO STEADY STATE FLOW

The discussion in paragraph 3.3 above voluntarily has been limited to the "general case" of steady state flow, ie. it corresponds to exclusive hypotheses for two special cases which we are now going to study, and which are that of isentropic flow of a constant composition gas on the one hand, and that of any steady state flow of a variable composition gas receiving energy (either directly in the form of heat, or following a chemical reaction) in a constant cross-section pipe on the other hand.

3.4.1 Special Case for Isentropic Flow of Constant Composition Gas

In this case, equation (3) for energy conservation is integrated immediately, as we know, to yield:

$$(25) \quad \frac{W^2 - W_0^2}{2} = H_0 - H.$$

In addition, equation (1) is integrated, as we know, even more readily, on the condition of assuming that the ratio of the principal specific heats of the fluid may be considered as constant, ie. that we may represent the isentropic expansion by the Poisson formula:

$$\frac{P}{\rho^\gamma} = C^{te},$$

which yields:

$$(26) \quad \frac{W^2 - W_0^2}{2} = \frac{\gamma}{\gamma - 1} \frac{R}{M} T_0 \left[1 - \left(\frac{P}{P_0} \right)^{\frac{\gamma - 1}{\gamma}} \right].$$

Formula (26) totally accounts for variations of kinetic energy as a function of pressure differences, and the temperature variations themselves are perfectly defined as a function of these same pressure variations, and accounted for by the Poisson formula through the well known relationship:

$$\frac{T}{T_0} = \left(\frac{P}{P_0} \right)^{\frac{\gamma - 1}{\gamma}}.$$

However, these formulae do not yield any precise data on the nature of the cross-section variations as a function of the pressure variations or vice versa; however, it suffices to refer to equation (11) to observe that it reduces in the case considered to:

$$(11') \quad \frac{dW}{W} = \frac{-a^2}{a^2 - W^2} \frac{d\Sigma}{\Sigma},$$

which clearly demonstrates that the fluid velocity effectively increases only for decreasing cross-sections, for fluid velocities less than the "speed of sound", for increasing cross-sections, for velocities greater than the "speed of sound" and that the cross-section must pass through a minimum for $W=a$, as demonstrated previously by Laval.

We also verify that equation (14) reduces in the case considered, and regardless of the cross-section of the tube considered and the corresponding divergence or convergence, to $n=\gamma$, which returns to the Poisson law [formula (14) being valid for any fluid obeying the Mariotte-Gay-Lussac law and having constant specific heats].

If we assume a priori that the tube is first convergent, then divergent (Laval or venturi tube), we see that:

-if the velocity W constantly remains less than the "speed of sound", it will begin by increasing in the convergent portion, then decreasing in the divergent portion, conforming to equation (11'), the pressure commencing by decreasing in the convergent portion and once again increasing in the divergent portion, conforming to formula (26), such that if the speed of sound is not attained in the throat of the tube, it cannot be reached subsequently;

-if on the contrary the velocity exceeds the value corresponding to the "speed of sound", we see that it must attain this in the throat of the tube and that, conforming to formula (11'), it continues to increase in the divergent portion of the tube, such that the pressure never stops decreasing and the velocity never stops increasing throughout the entire length of the tube.

Of course, it is easy to define the boundary conditions which separate the two flow regimes described above.

If the "speed of sound" is reached in the throat of the tube, the corresponding temperature T_c obviously is defined by equation (25) where we may assume that W_0 is negligible, which yields:

$$H_0 - H_c = C(T_0 - T_c) = \frac{1}{2} a^2 = \frac{\gamma R}{2M} T_c.$$

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or:

$$\frac{\gamma}{2} (C - c) T_c = C(T_0 - T_c)$$

where, by simplifying:

$$T(\gamma - 1 + 2) = 2T_0,$$

and:

$$\frac{T}{T_0} = \frac{2}{\gamma + 1},$$

or else:

$$\frac{P}{P_0} = \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma}{\gamma - 1}}.$$

Now, the sonic flow in the throat corresponds to the condition:

$$\mu = \rho_c \Sigma_c W_c = \rho_c \Sigma_c a = \rho_0 \frac{\rho_c}{\rho_0} \Sigma_c \gamma \frac{R}{M} T_0 \frac{T_c}{T_0},$$

or:

$$\mu = \frac{\mu}{\Sigma_0 W_0} \left(\frac{2}{\gamma + 1} \right)^{\frac{1}{\gamma - 1}} \Sigma_c \gamma \frac{R}{M} T_0 \frac{2}{\gamma + 1},$$

where:

(27)

$$W_0 = \frac{\Sigma_c}{\Sigma_0} \gamma \frac{R}{M} T_0 \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma}{\gamma - 1}},$$

or:

(28)

$$\mu = P_0 \Sigma_c \gamma \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma}{\gamma - 1}}$$

The desired condition necessary and sufficient for the "speed of sound" to be attained in the throat of the tube and for the supersonic regime to be "initiated" in this thus is that the velocity W or the flow μ may not be less than the values defined by formulae (27) and (28), the value of μ obviously capable of varying for the same initial temperature T_0 of the fluid, depending on the value of the upstream pressure P_0 .

Obviously it is useless to pursue the study of the special case of laminar isentropic and steady state flow in the case of a constant composition gas, since this study in reality has no other objective than to note the possibility of rediscovering known details of this flow from the general formulae established above.

3.4.2 Special Case for Steady State Flow With Heat Addition For Variable Composition Gases in Constant Cross-Section Tubes

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Within the same framework as for the study of the general case, we will begin by studying the nature of the flow in order to then proceed to the effective determination of the characteristics of the gas and its velocity in each cross-section.

3.4.2.1 Differential Study of Flow Velocity

3.4.2.1.1 Study of Velocity Gradient

In the special case considered, equation (11) simplifies to yield:

$$(11") \quad \frac{dW}{W} = \frac{a^2}{a^2 - W^2} \left[\frac{dQ - dE}{CT} - \frac{dM}{M} \right]$$

Considering that as a general rule the term $\frac{dQ - dE}{CT}$, representing the energy addition largely overrides the term $\frac{dM}{M}$ pertaining to the variation of mean molecular mass, we see in equation (11"):

-that a positive energy addition increases the velocity of the fluid, when this is less than the "speed of sound", and decreases it when it is greater than the "speed of sound";

-that the "speed of sound" cannot be attained unless the steady state condition ceases to exist except in a cross-section where the energy addition is zero (or more rigorously the quantity $\frac{dQ - dE}{CT} - \frac{dM}{M}$). If this

was not true, two shock waves of opposite directions would be produced, as we have demonstrated in the first part pertaining to variable state conditions, which would destroy the steady state character of the motion, so that at least this motion would be accelerated or decelerated depending on the precise relationship, also defined in the first part:

$$\frac{\partial W}{\partial t} = a^2 \left(\frac{1}{M} \frac{dM}{dt} + \frac{W}{\Sigma} \frac{\partial \Sigma}{\partial s} \right) - \frac{\gamma - 1}{W} \frac{dQ - dE}{dt};$$

-that the fluid velocity can only continue to increase beyond the "speed of sound" if in the corresponding region of the tube there is energy subtraction (or more precisely a negative value for the quantity $\frac{dQ - dE - dM}{CT M}$);

-in addition, that we may transform equation (11") by multiplying both terms by W^2 (to make the differential for the kinetic energy appear) and by expressing this quantity from equation (2), which yields:

$$(11''') \quad d\left(\frac{W^2}{2}\right) = \frac{1}{1 - \frac{W^2}{a^2}} \frac{\mu^2}{\rho^2 \Sigma^2} \left[\frac{dQ - dE}{CT} - \frac{dM}{M} \right].$$

We see from this equation that for the same value of the Mach number W , the increase in kinetic energy corresponding to a given energy addition is practically proportional to $\frac{\mu^2}{\rho^2 \Sigma^2}$, which allows us to believe that this increase in kinetic energy is a rapidly increasing function of the mass flow per unit cross-section $\frac{\mu}{\Sigma}$, with equality of the Mach number. /38

3.4.2.1.2 Study of Polytropic Gradient Present in Each Cross-Section

In the special case considered, formula (14) obviously reduces to:

$$n = \frac{\frac{dP}{P}}{\frac{d\rho}{\rho}} = \gamma \frac{W^2}{a^2}.$$

such that:

$$(29) \quad \frac{dT}{T} = \left(1 - \frac{1}{n}\right) \frac{dP}{P} + \frac{dM}{M}.$$

It turns out that the fluid flow commences by being substantially isobaric, corresponds to a polytropic coefficient which increases proportionally to the Mach-Sarrau number, becomes isothermic for $W = \frac{a}{\sqrt{\gamma}}$ if the variation of mean molecular mass $\frac{dM}{M}$ is zero or negligible, becomes adiabatic for $W=a$, and corresponds to a polytropic exponent greater than γ (ie. with expansion including a more rapid temperature drop than with isentropic expansion), while W becomes greater than a .

3.4.2.1.3 Study of Temperature Variation

It clearly could be deduced from the variations of the polytropic coefficient, if we can absolutely neglect the variation of mean molecular mass, that under these conditions the temperature would commence by increasing, would attain its maximum for $W = \frac{a}{\sqrt{\gamma}}$ and would then decrease more and more rapidly while W would continue to increase, ie. up to the speed of sound, if the energy transferred to the fluid is sufficient for this to be attained, and even beyond this, if the energy is provided to the fluid in the remainder of its flow, in such a manner that this could become readily supersonic.

In reality, and in a more precise fashion, we see in equation (9), by setting $d\Sigma=0$, that the temperature maximum is attained for the value $\frac{W}{a}$, slightly different from $\frac{1}{\sqrt{\gamma}}$ defined by the relationship:

$$\text{or: } T \frac{dM}{M} = -\frac{M}{R} (dQ - dE) \left(\frac{a^2}{\gamma W^2} - 1 \right),$$

$$\frac{a^2}{W^2} = \gamma - \frac{a^2}{M} \frac{dM}{dQ - dE}.$$

3.4.2.1.4 Pressure Variations Throughout Tube

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Equation (24) obviously is simplified and integrated for a constant cross-section Σ , yielding the following relationship, deducible from a direct application of the theorem for projected quantities of motion:

$$(24') \quad P_0 - P = \frac{\mu}{\Sigma} (W - W_0).$$

This equation demonstrates in a clear fashion that in a constant cross-section tube, the pressure varies linearly as a function of the velocity and in opposite direction from this.

It also demonstrates that the pressure difference required to produce an increase in velocity $W - W_0$ increases proportionally to the mass-flow μ per unit of cross-section and this obviously has considerable Σ practical significance.

This last conclusion must be reconciled with what has been stated at the end of paragraph 3.4.2.1.1, and according to which the increase in kinetic energy is an increasing function of the mass-flow per unit of cross-section μ , with equality of Mach numbers. From that, this

mass-flow per unit of cross-section now would appear as a favorable factor in the obtainment of a large kinetic energy for a small energy addition during the expansion of the fluid and an unfavorable factor in the obtainment of a large kinetic energy for a small pressure difference. The direct calculation of the velocity W produced will

confirm and precisely define this point of view, as we will see.

3.4.2.2 Calculation of Velocity Produced in Each Cross-Section

It is easy, in the special case considered, to calculate directly the velocity W produced in a given cross-section, and this regardless of the physical state of the fluid in the tube inlet cross-section.

In fact, if we designate the total quantity of heat received per unit of mass of the fluid by Q , from the upstream cross-section corresponding to the initial conditions $P_0, \rho_0, T_0, \Sigma_0, H_0, E_0$ and W_0 up to the cross-section considered and the potential chemical energy corresponding to the state of the fluid in this second cross-section by E , we may write:

$$\frac{W^2 - W_0^2}{2} = Q + (E_0 - E) + H_0 - H,$$

such that by considering:

$$P_0 - P = \frac{\mu}{\Sigma} (W - W_0),$$

$$\mu = \rho \Sigma W = \rho_0 \Sigma W_0,$$

and:

$$\frac{P}{\rho} = \frac{R}{M} T$$

(if we assume that the given cross-section considered is sufficiently far from the inlet cross-section so that the fluid has become entirely gaseous) it becomes:

$$T = \frac{M}{R} \frac{P_0 - \frac{\mu}{\Sigma} (W - W_0)}{\frac{\mu}{\Sigma} W},$$

where:

$$\frac{W^2 - W_0^2}{2} = Q + E_0 - E + H_0 - C \frac{M \Sigma W}{R \mu} \left[P_0 - \frac{\mu}{\Sigma} (W - W_0) \right],$$

or:

$$\frac{W^2 - W_0^2}{2} = Q + E_0 - E + H_0 - \frac{\gamma}{\gamma - 1} \frac{\Sigma W}{\mu} \left[P_0 - \frac{\mu}{\Sigma} (W - W_0) \right],$$

or even:

$$W^2 \left(\frac{1}{2} - \frac{\gamma}{\gamma - 1} \right) + \frac{\Sigma}{\mu} \frac{\gamma}{\gamma - 1} W \left[P_0 + \frac{\mu}{\Sigma} W_0 \right] - Q - (E_0 - E) - H_0 - \frac{W_0^2}{2} = 0.$$

From this, we may suggest:

$$E_0 = H_0 + \frac{W_0^2}{2}$$

(initial potential energy of the fluid), and:

$$E = Q + E_0 - E$$

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(energy provided to the fluid between its entry into the tube and the section considered).

With these notations, the general equation yielding the velocity W in the case considered for a steady state flow in a cylindrical tube assumes the simple form:

$$(30) \quad \frac{\gamma+1}{\gamma-1} \frac{W^2}{2} + \frac{\gamma}{\gamma-1} \frac{\Sigma}{\mu} \left(P_0 + \frac{\mu}{\Sigma} W_0 \right) W + \mathcal{E}_0 + \mathcal{E} = 0,$$

an equation in which it should be noted that the quantity $P_0 + \frac{\mu}{\Sigma} W_0$, which defines the conditions for entry of the fluid into the tube, is an unvarying value which is preserved throughout the entire tube and which we may represent for simplicity by π .

Equation (30) being a second order equation (for reasons which will be given later), it obviously only has a solution if its discriminant Δ is positive, ie. if:

$$\Delta = \frac{1}{\gamma-1} \left[\gamma^2 \frac{\Sigma^2}{\mu^2} \pi^2 - 2(\gamma+1)(\mathcal{E}_0 + \mathcal{E}) \right] \geq 0,$$

or:

$$(31) \quad \mathcal{E}_0 + \mathcal{E} \leq \frac{\gamma^2 \Sigma^2 \pi^2}{2(\gamma+1)\mu^2}.$$

We already see with condition (31) that the flow is not possible under steady state conditions unless the total quantity of energy $\mathcal{E}_0 + \mathcal{E}$ is not too high with respect to the value of the unvarying quantity defining the initial conditions for the flow:

$$\pi = P_0 + \frac{\mu}{\Sigma} W_0.$$

Inversely, we also may state that the flow is only possible under steady state conditions if the value of the unvarying quantity π is sufficiently high with respect to the total energy $\mathcal{E}_0 + \mathcal{E}$.

We will better comprehend the actual meaning of the relationship defined between π and $(\mathcal{E}_0 + \mathcal{E})$ through the condition of positive determinant if we once again state the problem, as we have constantly done during the present study, by assuming that the Mariotte-Gay-Lussac-Avogadro law is applicable to the fluid (assumed to be gaseous after its entry into the tube).

With this hypothesis, we may in fact write:

$$(\gamma-1)\mathcal{E}_0 = (\gamma-1) \frac{W_0^2}{2} + C(\gamma-1)T_0 = (\gamma-1) \frac{W_0^2}{2} + \gamma \frac{R}{M} T_0 = (\gamma-1) \frac{W_0^2}{2} + \gamma \frac{\Sigma}{\mu} P_0 W_0.$$

Such that equation (30) becomes:

$$(32) \quad (\gamma+1)W^2 - 2\gamma \frac{\Sigma}{\mu} \left(P_0 + \frac{\mu}{\Sigma} W_0 \right) W + 2(\gamma-1) \times \left[Q + \mathcal{E}_0 - \mathcal{E} + \frac{\gamma}{\gamma-1} \frac{\Sigma}{\mu} P_0 W_0 + \frac{W_0^2}{2} \right] = 0.$$

The discriminant corresponding to this new form of the equation thus is written:

$$\Delta' = \gamma^2 \left(\frac{\Sigma P_0}{\mu} + W_0 \right)^2 - 2(\gamma + 1)(\gamma - 1) \left(Q + E_0 - E + \frac{\gamma}{\gamma - 1} \frac{\Sigma}{\mu} P_0 W + W_0^2 \right),$$

or:

$$\Delta' = \gamma^2 \frac{\Sigma^2 P_0^2}{\mu^2} - 2\gamma \Sigma \frac{P_0}{\mu} W_0 + W_0^2 - 2(\gamma^2 - 1)(Q + E_0 - E),$$

or else:

$$\Delta' = \left(\gamma \frac{\Sigma P_0}{\mu} - W_0 \right)^2 - 2(\gamma^2 - 1)(Q + E_0 - E),$$

or even, by designating the "speed of sound" by a_0^2 in the inlet cross-section:

$$\Delta' = \left(\frac{a_0^2}{W_0} - W_0 \right)^2 - 2(\gamma^2 - 1)(Q + E_0 - E),$$

such that the condition for reality of the roots of equation (32), and thus the possibility of steady state flow, may be written either as:

$$(33) \quad \kappa \leq \frac{1}{W_0^2} \frac{(a_0^2 - W_0^2)^2}{2(\gamma^2 - 1)},$$

or:

$$(34) \quad \left(\gamma \frac{\Sigma}{\mu} \rho_0 \frac{R}{M} T_0 - \frac{\mu}{\Sigma \rho_0} \right)^2 \geq 2(\gamma^2 - 1) \kappa.$$

We see on both equation (33) and equation (34) that for a sufficiently small mass-flow per unit of cross-section $\frac{\mu}{\Sigma}$ or, which turns out to be

the same, for a sufficiently small initial velocity W_0 at the tube inlet, the steady state flow in a cylindrical tube will still be possible, even if the total energy \mathcal{E} is considerable.

3.4.2.2.1 Discussion of Possibility of Condition of Steady State Flow With Constant Cross-Section Tubes

According to the discussion above on the subject of the variations of the velocity W , it is easy to account for the fact that we never may add a total quantity of energy to the fluid greater than that which is precisely necessary to bring the velocity of the fluid to the value a corresponding to the "speed of sound" in the cross-section considered. In fact, we have seen that once the "speed of sound" $W=a$ is attained, it was no longer possible to increase the velocity of the fluid except by taking energy from it and that we could no longer transfer energy to it for velocities less than a without increasing its kinetic energy, such that the algebraic sum of the quantities of energy which we may transfer to the fluid cannot be less than or equal to that which brings the velocity to its value $W=a$.

This deduction is also easy to verify with equation (32).

In fact, the condition $\Delta \geq 0$ is satisfactory, as we have seen from formulae (33) and (34), for an even greater total energy when Δ is smaller, such that \mathcal{E} is maximum (for a given unvarying value π) for $\Delta=0$, ie. when equation (32) has a double root.

The value of this double root may be calculated easily by writing:

$$W = \frac{\gamma \left(\frac{\Sigma}{\mu} P_0 + W_0 \right)}{\gamma + 1} = \frac{\gamma}{\gamma + 1} \left(\frac{P_0}{\rho_0 W_0} + W_0 \right) = \frac{\gamma}{\gamma + 1} \left(\frac{P_0}{\rho W} + W_0 \right)$$

where, considering the relationship:

$$P_0 - P = \frac{\mu}{\Sigma} (W - W_0),$$

$$(\gamma + 1) W = \frac{\gamma P}{\rho W} + \frac{\mu}{\Sigma} \frac{W - W_0}{\rho W} + W_0;$$

so that:

$$\frac{\mu}{\Sigma \rho W} = 1,$$

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such that we simply have:

$$(\gamma + 1) W = \frac{\gamma P}{\rho W} + W,$$

or:

$$W^2 = \frac{\gamma P}{\rho} = a^2.$$

Thus it is verified that if the quantity of energy which may be transferred to the fluid for the given values of $\frac{\mu}{\Sigma}$, P_0 and W_0 , or more simply for the unvarying π , is limited, because this quantity cannot exceed the value for which W becomes equal to the speed of sound a (even if W can be raised considerably above this value).

3.4.2.2.2 Selection of Root Corresponding to Actual Flow

From the preceding discussion, it is obvious that with equations (30) and (31) assumed to have their actual roots, the smallest of these is attained if the positive quantity of energy \mathcal{E} is provided to the fluid without being exceeded, while the largest is attained when we first provide to the fluid the quantity of energy just sufficient to reach the sonic velocity $W=a$ [ie. that which defined equations (33) and (34)] and when we then remove the quantity of energy required so that the algebraic sum of the quantities provided is equal to the value \mathcal{E} which defines the two roots considered.

We have seen that it is indispensable if the speed of sound may be exceeded, without motion ceasing to be steady state and without the value π of the unvarying quantity $(\sum P_0 + W_0)$ being modified, that the energy addition is cancelled and changes sign when W passes the value a .

An additional verification, due to the fact that the smallest root of equations (30) and (31) is the only one to be considered when the energy is added constantly until we attain a specific value \mathcal{E} , but without exceeding it, also may be derived from the fact that for an infinitely small energy addition, we obviously have $W=W_0$. In fact, this is found to be precisely true, as we will see, for the smaller of the two roots, when we set $\mathcal{E}=0$.

It is also obvious that as the energy transferred to the fluid increases progressively from 0 to \mathcal{E} , it is always the same root which yields the actual velocity of the flow, and that we may not pass from one root to the other unless we attain in the actual flow the quantity of energy which corresponds to the double root $W=a$.

In a more precise fashion, we may place the roots of equation (32) into the form: /44

$$W = \frac{\gamma \left(\frac{\sum}{\mu} P_0 + W_0 \right) \pm \sqrt{\left(\gamma \frac{\sum}{\mu} P_0 - W_0 \right)^2 + \mathcal{E}}}{\gamma + 1}$$

such that the smallest root assumes the following form for $\mathcal{E}=0$:

$$W_1 = \frac{\gamma W_0 + W_0}{\gamma + 1} = W_0$$

3.4.2.2.3 Ratio Between Increase of Kinetic Energy Obtained, Energy Transferred to Fluid and Mass-Flow Unit

It suffices to note that equation (11') may be written:

$$d\left(\frac{W^2}{2}\right) = \frac{a^2}{a^2 - W^2} W^2 \left[\frac{dQ - dE}{CT} - \frac{dM}{M} \right] = \frac{a^2}{a^2 - W^2} \frac{1}{\rho^2 \Sigma^2} \left[\frac{dQ - dE}{CT} - \frac{dM}{M} \right]$$

to predict that under subsonic conditions, the energy added to the fluid contributes even more to increase its kinetic energy when the gas velocity has a higher value, and that consequently the total increase in kinetic energy must represent an even smaller fraction of the energy transferred to the fluid, all other things being equal, when the unit flow $\frac{\mu}{\Sigma}$ itself is small.

We have seen already, and it has been verified, that under supersonic conditions it is on the contrary by deducting energy from the fluid that we may increase the velocity (this deduction also contributing to increase the kinetic energy of the fluid with a coefficient whose value

deviates from infinity for $W=a$ and decreases as W increases).

In summary, we see that the energy transferred to the fluid during its flow has an influence on its kinetic energy, which depends on both the mass flow $\frac{\mu}{\Sigma}$ and the Mach number $\frac{W}{a}$ corresponding to the cross-section considered and that this influence is even more intense when the mass-flow per unit of cross-section is higher (with equality for the Mach number), and that the Mach number realized at the moment considered in the fluid flow is closer to one.

3.4.2.3 Influence of Mass-Flow Unit and Mach Number on Pressure Difference and Density

Equation (25') already demonstrates that the difference in pressures existing between the two ends of the tube must increase rapidly with the unit flow $\frac{\mu}{\Sigma}$, given that the velocities W and W_0 also cannot fail to increase with this flow.

In a more precise fashion, we also may transform equation (25') by considering the expressions for flow so as to place it in the form:

$$(25'') \quad P_0 - P = \frac{\mu^2}{\Sigma^2} \left(\frac{1}{\rho} - \frac{1}{\rho_0} \right).$$

This demonstrates that the difference in pressures in question must increase more rapidly than the square of the unit flow, the density difference $\rho_0 - \rho$ obviously increasing during the expansion, such that also increases.

It also remains true that the maximum velocity attained by the fluid is less than the "speed of sound" or that the supersonic conditions are found to occur on the downstream side of the tube, through subtraction of heat or energy from the fluid. In fact, we may observe that at any instant, n being the exponent of the polytropic tangent, we have:

$$\frac{d\rho}{\rho} = \frac{1}{n} \frac{dP}{P}$$

and, since the velocity can increase only on the condition that the pressure gradient $\frac{\partial P}{\partial s}$ is negative, by virtue of equation (24) $dP = -\frac{\mu}{\Sigma} dW$,

we see that ρ decreases constantly along the flow, with the single condition that the velocity itself increases constantly (since the exponent n of the polytropic tangent is always positive).

Considering the two preceding equations and the expression for the polytropic coefficient previously obtained:

$$n = \gamma \frac{W^2}{a^2};$$

we also may obtain a relationship which more perfectly defines the variations of the density during the flow, by writing:

$$(35) \quad \frac{1}{\rho} \frac{d\rho}{dW} = -\frac{1}{n} \frac{\mu}{\Sigma} = -\frac{\mu}{\Sigma} \frac{a^2}{W^2}.$$

We see in formula (35) that the derivative for the relative variation of density with respect to the velocity $\frac{1}{\rho} \frac{d\rho}{dW}$ is both proportional to the mass-flow $\frac{\mu}{\Sigma}$ and inversely proportional to the Mach number corresponding to the cross-section or to the flow phase considered.

We also may express more directly the density variation as a function of the velocity variation by juxtaposing (25') and (25'') in the form:

$$P_0 - P = \frac{\mu}{\Sigma} (W - W_0) = \frac{\mu^2}{\Sigma^2} \left(\frac{1}{\rho} - \frac{1}{\rho_0} \right),$$

such that:

$$(36) \quad \frac{1}{\rho} - \frac{1}{\rho_0} = \frac{\Sigma}{\mu} (W - W_0)$$

(relationship which is derived directly from the continuity equation $\mu = \rho \Sigma W$, μ and Σ assumed to be constant), which demonstrates that the difference of the inverses of the specific downstream and upstream volumes is proportional to the unit mass-flow, with equality of velocity increase. /46

With respect to the pressure difference, we may note that it may be placed in the form:

$$-(P_0 - P) = \frac{\mu}{\Sigma} W_0 \left(\frac{W}{W_0} - 1 \right),$$

which demonstrates that, with equality of the Mach number, it is proportional to $\frac{\mu}{\Sigma W_0}$, thus to the density ρ_0 present in the upstream section.

3.4.2.4 Comparison Between Sonic Velocity Obtained With Cylindrical Tubes and Sonic Velocity Obtained Through Isobaric Combustion Followed by Isentropic Expansion

In the case of the cylindrical tube, we see in equation (30) that the product of the roots, thus the square of the double root, has the value:

$$W_1 W_2 = W^2 = 2 \frac{\gamma - 1}{\gamma + 1} (\kappa + \kappa_0).$$

In the case of isobaric combustion followed by isentropic expansion, we also have, for the same initial temperature T_0 and for the same total energy provided to the fluid \mathcal{C} , a combustion temperature:

$$T_0 + \frac{\mathcal{C}}{C}.$$

The temperature at the throat, after isentropic expansion, thus will have the value:

$$\left(T_0 + \frac{\epsilon}{C}\right) \frac{2}{\gamma + 1}.$$

The fluid velocity at the throat, after isobaric combustion and isentropic expansion, thus will have the value:

$$a^2 = \gamma \frac{R}{M} \left(T_0 + \frac{\epsilon}{C}\right) \frac{2}{\gamma + 1} = 2 C \frac{\gamma - 1}{\gamma + 1} \left(T_0 + \frac{\epsilon}{C}\right) = 2 \frac{\gamma - 1}{\gamma + 1} \left(\epsilon + \epsilon_0 - \frac{W_0^2}{2}\right).$$

The "sonic velocity" produced from the same initial temperature of the fluid and for the same total energy addition, for a cylindrical tube and for isobaric combustion followed by isentropic expansion, thus is the same if we assume that the initial kinetic energy $\frac{W_0^2}{2}$ is

negligible in the case of the cylindrical tube (given that it is assumed to be zero with isobaric combustion followed by isentropic expansion).

For the same reasons, the temperatures occurring in the sonic cross-sections are substantially the same for the same initial temperature T_0 in a cylindrical tube and with isobaric combustion followed by isentropic expansion.

At first glance, this result may be surprising since the temperature attained during the transformation is lower in the first case than in the second, but we will see on the contrary that the upstream pressure is lower with isobaric combustion followed by isentropic expansion than with expansion with combustion in cylindrical tubes.

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3.4.2.5 Comparison Between Pressures

3.4.2.5.1 Comparison Between Upstream Pressures Producing the Same Sonic Velocity With Cylindrical Tubes and With Isobaric Combustion Followed by Isentropic Expansion

In the case of the cylindrical tube, we obviously have:

$$(37) \quad P_0 - P = \frac{\mu}{\Sigma} \left[\sqrt{2 \frac{\gamma - 1}{\gamma + 1} (\epsilon + \epsilon_0)} - W_0 \right],$$

an expression in which $\epsilon + \epsilon_0$ is determined by the double root condition:

$$\epsilon + \epsilon_0 = \frac{\gamma^2 \Sigma^2}{2(\gamma + 1) \mu^2} \left(P_0 + \frac{\mu}{\Sigma} W_0 \right)^2.$$

On the other hand, with isobaric combustion followed by isentropic expansion, we have:

$$\frac{P_0}{P} = \left(\frac{\gamma + 1}{2} \right)^{\frac{\gamma}{\gamma - 1}},$$

where:

$$(38) \quad P_0 - P = P_0 \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma}{\gamma - 1}} \left[\left(\frac{\gamma + 1}{2} \right)^{\frac{\gamma}{\gamma - 1}} - 1 \right] = P_0 \left[1 - \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma}{\gamma - 1}} \right].$$

In principle, formulae (37) and (38) permit us to compare the upstream pressures corresponding to the two cases considered, but it is not necessary to discuss them in order to be able to confirm that the upstream pressure is stronger in the case of the cylindrical tube for the same pressure P in the downstream sonic cross-section.

In fact, we have:

$$P_0 - P = \mu \int_{\Sigma} \frac{1}{\Sigma} dW$$

in both cases, and since the cross-sections Σ corresponding to the same terminal cross-section at the throat inevitably are stronger in the case of isentropic expansion in a convergent tube, the difference $P_0 - P$ corresponding to the same final sonic velocity W (and necessarily to the same initial velocity W_0) is inevitable lower for this type of tube, all other things being equal.

Finally, we see that while the initial temperatures resulting in substantially identical sonic velocities are the same for both expansions, the initial pressures are different, and consequently the initial entropies as well.

This last point permits a better understanding of how the same energy addition results in substantially the same increase in kinetic energy for two thermodynamic flows having the same initial temperature and nearly the same final temperatures, and very different maximum temperatures.

The entropic representation of these flows on the same diagram in figure 2 also demonstrates that the areas underlying the two corresponding curves may be very similar, despite the notable differences of the maximum temperatures in question.

3.4.2.5.2 Ratio Between Upstream Pressures Corresponding to the Same Propulsion Force Per Unit of Sonic Cross-Section

If we consider a cylindrical tube and a convergent tube with isentropic expansion, both operating under sonic conditions and having the same sonic cross-section, we can calculate easily the ratio of the corresponding upstream pressures.

Esnault-Pelleterie has established in fact that for a tube with isentropic expansion, limited to its convergent portion, the static propulsion force is:

$$F = \Sigma_c (\gamma + 1) P_c.$$

P_c being the pressure at the throat and Σ_c the corresponding cross-section.

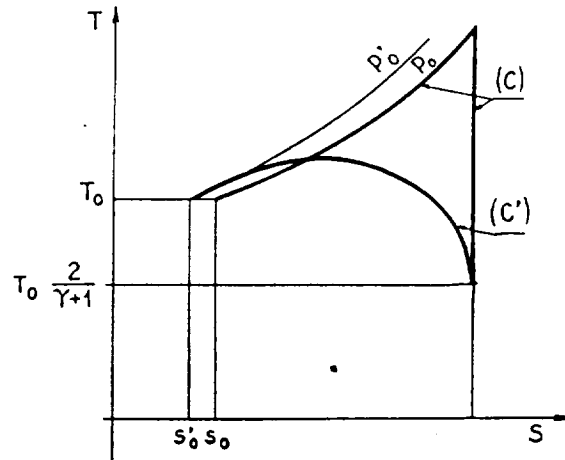


Figure 2.

- (C) Isobaric, then adiabatic flow.
 (C') Flow occurring in a cylindrical tube for a final temperature and sonic velocity nearly identical to the final temperature and sonic velocity of (C).

Thus, in this case the upstream pressure has the following value:

$$P = P_c \left(\frac{\gamma + 1}{2} \right)^{\frac{\gamma}{\gamma - 1}} = \frac{F}{\Sigma_c \gamma + 1} \left(\frac{\gamma + 1}{2} \right)^{\frac{\gamma}{\gamma - 1}}.$$

On the contrary, for the cylindrical tube, the upstream pressure obviously has the value:

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$$P_0 = \frac{F}{\Sigma}$$

For the same propulsion force value F and for $\Sigma = \Sigma_c$, we thus have:

$$(39) \quad \frac{P_0}{P} = (\gamma + 1) \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma}{\gamma - 1}} = \left(\frac{1}{\gamma + 1} \right)^{\frac{1}{\gamma - 1}} (2)^{\frac{\gamma}{\gamma - 1}}.$$

Numerically, the pressure ratio in question obviously depends on the value of γ .

By setting $\gamma = 1.3$, we find:

$$\frac{P_0}{P} = 1.26$$

3.4.2.6 Thermal and Energy Outputs Obtained With Each Cross-Section

To obtain a value for the ratio existing between the increase in kinetic energy and the energy $dQ-dE=d$ provided to the fluid, it suffices to observe that in the case considered for steady state flow in a cylindrical tube, we have on the one hand:

$$n = \gamma \frac{W^2}{a^2}$$

or:

$$\frac{a^2}{W^2} = \frac{\gamma}{n}$$

and on the other hand, we may write:

$$\frac{1}{CT} = \frac{1}{C} \frac{R}{M} \frac{\rho}{P} = \frac{\gamma-1}{\gamma} \frac{\rho}{P} = \frac{\gamma-1}{a^2}.$$

Considering these relationships, equation (11) in fact may be placed into the following form:

$$(40) \quad \left(\frac{\gamma}{n} - 1 \right) d \left(\frac{W^2}{2} \right) = (\gamma - 1) d \varepsilon - a^2 \frac{dM}{M}.$$

It then is sufficient to recall that n increases constantly with W (by virtue of the expression above), and we see that the "instantaneous thermal yield", defined by the ratio of the energies in question, would be zero for $W=0$ if $\frac{dM}{M}$ was negligible and with the same hypothesis;

$\frac{dM}{M}$ increases constantly from zero to more than infinity when W increases from zero to a .

If we neglect the variation of molecular mass $\frac{dM}{M}$, in fact we obtain for the expression estimating the yield in question:

$$(41) \quad \mathcal{R}_t = \frac{d \left(\frac{W^2}{2} \right)}{d \varepsilon} \approx \frac{\gamma-1}{\frac{\gamma}{n}-1} = \frac{\gamma-1}{\frac{a^2}{W^2}-1}.$$

If W exceeds the "speed of sound", we have seen that it becomes necessary to subtract energy to obtain an increase of velocity and kinetic energy in the fluid, such that it becomes absurd to speak of an "instantaneous thermal yield" corresponding to the definition above.

In any case, it must be recalled that besides the energy provided to the fluid during its flow, on the upstream side it receives energy $\frac{P_0}{\rho_0}$ and on the downstream side returns energy to the medium equal to $\frac{P}{\rho}$.

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Considering this remark, it is logical to consider an "instantaneous thermal yield", in the denominator of which there appears the sum of the energies provided to the fluid in the form of heat and work, and which consequently can be written:

$$\mathcal{R}_e = \frac{d\left(\frac{W^2}{2}\right)}{d\mathcal{E} - d\left(\frac{P}{\rho}\right)}.$$

Noting that $\frac{P}{\rho} = \frac{1}{\gamma} a^2$, thus we have:

$$(42) \quad \mathcal{R}_e = \mathcal{R}_t \frac{d\mathcal{E}}{d\mathcal{E} - d\left(\frac{a^2}{\gamma}\right)} = \frac{\gamma-1}{\frac{a^2}{W^2} - 1} \frac{d\mathcal{E}}{d\mathcal{E} - \frac{1}{\gamma} d(a^2)},$$

or else, if we neglect the variation of mean molecular mass and consequently if we consider $\frac{R}{M}$ as a constant:

$$(42') \quad \mathcal{R}_e \approx \frac{\gamma-1}{\frac{\gamma}{n}-1} \frac{d\mathcal{E}}{d\mathcal{E} - \frac{R}{M} dT}.$$

We see in formulae (42) and (42') that the "instantaneous energy yield" becomes less than the "instantaneous thermal yield" from the moment when the temperature of the fluid has exceeded its maximum (ie. as we have seen and will see later, from the moment that W exceeds the value $a\sqrt{\gamma}$ in the case where the variation of mean molecular mass is negligible), while it is higher than the "instantaneous thermal yield" for values of W comprised between zero and the critical value in question. a

For $W=a$, we also have $n=\gamma$, and $d\mathcal{E}$ must be zero so that the motion remains steady state, such that the value of the "instantaneous energy yield" can only be calculated by formulae (42) or (42'), but we may write directly:

$$d\left(\frac{W^2}{2}\right) = d\mathcal{E} - C dT$$

and consequently, if we still neglect $\frac{dM}{M}$:

$$\mathcal{R}_e \approx \frac{d\left(\frac{W^2}{2}\right)}{d\mathcal{E} - \frac{R}{M} dT} = \frac{d\mathcal{E} - C dT}{d\mathcal{E} - \frac{R}{M} dT} = \frac{C}{C-c} = \frac{\gamma}{\gamma-1}.$$

Finally, we see that, to the extent that we can neglect the term $\frac{dM}{M}$ pertaining to the variation of mean molecular mass with respect to $\frac{d\mathcal{E}}{dT}$, the "instantaneous energy yield" realized during steady state conditions in a cylindrical tube begins at a very low value for $W=W_0$ (theoretically zero for $W=W_0=0$), increases with W so as to attain the value of 1 for

and attains the value $\frac{\gamma}{\gamma-1}$ for $\frac{W}{a}=1$.

Thus it is possible to define algebraically the value for the instantaneous energy yield in question, on the condition that the variation of mean molecular mass can be neglected, and such that this yield is expressed only as a function of the Mach number $\frac{W}{a}$ reached in the cross-section considered.

If we neglect $\frac{dM}{M}$, equation (9) in fact may be written:

$$(42) \quad \frac{R}{M} dT \approx d\xi \frac{\gamma-1}{\gamma} \frac{a^2 - \gamma W^2}{a^2 - W^2}.$$

Thus, we have:

$$\mathcal{R}_e \approx \frac{n(\gamma-1)}{\gamma-n} \frac{d\xi}{d\xi \left[1 - \frac{\gamma-1}{\gamma} \frac{a^2 - \gamma W^2}{a^2 - W^2} \right]}.$$

And since we have found that $n = \gamma \frac{W^2}{a^2}$, after all reductions are made, it becomes:

$$(43) \quad \mathcal{R}_e \approx \frac{\frac{W^2}{a^2} (\gamma-1)}{\frac{1}{\gamma} + (\gamma-2) \frac{W^2}{a^2}}.$$

Formula (43) permits us to discover the values for the instantaneous energy yield above calculated for $W = \frac{a}{\sqrt{\gamma}}$ and for $W=a$. In addition, we see that said instantaneous energy yield increases with $\frac{W}{a}$ if the velocity W exceeds the value a (and theoretically would tend to approach $\frac{\gamma-1}{\gamma-2}$ for infinite $\frac{W}{a}$).

3.4.2.7 Temperature Variation Obtained During Flow

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We see that if we neglect the variation of mean molecular mass, we may place the temperature derivative with respect to the energy addition in the simplified form (42), which may be written:

$$(42') \quad \frac{dT}{d\xi} \approx \frac{1}{C} \frac{a^2 - \gamma W^2}{a^2 - W^2},$$

a relationship which confirms that the temperature increases with $\frac{W}{a}$ until W has attained the value $\frac{a}{\sqrt{\gamma}}$, that the derivative $\frac{dT}{d\xi}$ becomes infinite and negative for $W = a - \epsilon$, and infinite and positive for $W = a + \epsilon$, and that its absolute value then decreases when $\frac{W}{a}$ increases, if the conditions present permit the fluid to exceed the "speed of sound"

(which assumes, as we have seen, that it begins by having subtraction of energy).

In addition, the overall expression for temperature already has been calculated (see §3.4.2.2) to obtain the second degree equation which yields the values of W and may be placed into the form:

$$(44) \quad T = \frac{1}{C-c} W \left[\frac{\Sigma}{\mu} P_0 - (W - W_0) \right].$$

The temperature thus is known at any point of the flow, as well as the velocity and consequently also the density.

If we wish to determine the maximum value T_m for said temperature, and if we assume that we can neglect the influence of the variations of mean molecular mass, obviously it is sufficient to set $W = \frac{a}{\sqrt{\gamma}}$ in equation (44), which yields:

$$T_m = \frac{1}{C-c} \sqrt{\frac{R}{M}} T \left[\frac{\Sigma}{\mu} P_0 - \left(\sqrt{\frac{R}{M}} T - W_0 \right) \right],$$

or after all reductions have been performed, and since $T=T_m$:

$$T_m = \frac{1}{4(C-c)} \left[\frac{\Sigma}{\mu} P_0 + W_0 \right]^2.$$

This last formula obviously allows us to compare the maximum temperature occurring under steady state conditions in a cylindrical tube and the maximum temperature occurring for example with isobaric combustion followed by isentropic expansion.

In fact, if we consider the instant of passage through the "sonic velocity", we find the temperature T_s in the case of a cylindrical tube by inserting this value for the velocity into equation (44):

$$T_s = \frac{1}{C-c} \sqrt{\gamma \frac{R}{M}} T \left[\frac{\Sigma}{\mu} P_0 - \sqrt{\gamma \frac{R}{M}} T + W_0 \right],$$

where, since $T=T_s$:

$$\sqrt{T_s} = \frac{1}{C-c} \sqrt{\gamma \frac{R}{M}} \left[\frac{\Sigma}{\mu} P_0 - \sqrt{\gamma \frac{R}{M}} T_s + W_0 \right]$$

where finally:

$$T_s = \frac{1}{c} \frac{\gamma}{(\gamma-1)(1+\gamma)^2} \left(\frac{\Sigma}{\mu} P_0 + W_0 \right)^2.$$

Since obviously we only can compare flows yielding the same "sonic velocity" in a truly valid fashion, it is obvious that the ratio of the maximum temperatures to be compared has the value:

$$\frac{T_m}{T_s \frac{\gamma+1}{2}}.$$

Now, according to the preceding:

$$(45) \quad \frac{T_m}{T_s} = \frac{(\gamma + 1)^2}{4\gamma},$$

such that the maximum temperature attained in a cylindrical tube regulated to provide a given "sonic velocity" (ie. a given temperature T_s in the cross-section where this velocity is attained) on the one hand, and the maximum temperature required to produce the same "sonic velocity" in a tube with isentropic expansion on the other hand, are given by the ratio:

$$(46) \quad R = \frac{\gamma + 1}{2\gamma}.$$

It is useful to note that by virtue of formula (45), the ratio between the maximum temperature attained during steady state conditions in a cylindrical tube and the temperature occurring in the cross-sections where the "sonic velocity" is attained during the same flow, is independent of the characteristics of the fluid in the inlet cross-section of said tube (naturally, with the sole restriction that the energy addition and the initial characteristics in question must correspond to obtainment of a "sonic velocity").

4. CONCLUSIONS

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From the preceding study, in our opinion it is possible to directly formulate the following conclusions which we will attempt to complete with a last observation.

4.1 VARIABLE STATE CONDITIONS

4.1.1 Free Selection of Parameters for Velocities Other Than "Speed of Sound"

To the extent that the fluid flow can be considered as laminar, one-dimensional and free of viscosity, it appears that we may arbitrarily set, regardless of the cross-section considered, the mass-flow μ , the cross-section Σ and its gradient $\frac{d\Sigma}{ds}$, the possible addition of heat $\frac{\partial Q}{\partial s}$, the possible potential chemical variation $\frac{\partial \mathcal{C}}{\partial s}$, the corresponding variation of mean molecular mass $\frac{\partial M}{\partial s}$ and the acceleration of motion $\frac{\partial W}{\partial t}$, except possibly in the cross-section where the velocity of the fluid would be equal to the "speed of sound"

$$a = \sqrt{\gamma \frac{R}{M} T}.$$

4.1.2 Relationship Required for Flow to Retain Finite Velocity Gradient and Pressure Gradient Even in Cross-Section Where Fluid Attains "Speed of Sound"

In a cross-section where the fluid velocity has attained the critical value $a = \sqrt{\gamma \frac{R}{M} T}$, a precise and unique relationship must exist between the different parameters considered above such that the velocity gradient and pressure gradient retain finite values (or if we prefer, such that the representative curve for these gradients as a function of the abscissa do not display a reflection point with tangent vertical to said cross-section), said relationship being written:

$$\frac{1}{\gamma-1} \frac{\mu}{\rho \Sigma} \frac{\partial W}{\partial t} + \frac{dQ - dE}{dt} - \frac{a^2}{\gamma-1} \frac{1}{M} \frac{dM}{dt} - \frac{a^2}{\gamma-1} \frac{\mu}{\rho \Sigma^2} \frac{d\Sigma}{ds} = 0,$$

or more simply:

$$\frac{\mu}{\rho \Sigma} \frac{\partial W}{\partial t} + (\gamma-1) \frac{dQ - dE}{dt} - a^2 \left(\frac{1}{M} \frac{dM}{dt} + \frac{1}{\Sigma} \frac{d\Sigma}{dt} \right) = 0.$$

While the relationship in question is not satisfactory, the peculiarity which appeared for the cross-section considered in the representative curve for the velocity gradients produces an analogous peculiarity for the representative curve for pressures as a function of the abscissa, from which comes the appearance of two shock waves of opposite directions which begin in said cross-section and are propagated toward the ends of the tube and modify the boundary conditions (in a manner which obviously depends on the conditions under which the tube is supplied with gas and those under which the gases which enter are evacuated and which determine the subsequent development of the phenomenon).

4.1.3 Sign of Velocity Gradient as Function of Principal Parameters for $W < a$ or $W > a$

In the regions where the fluid velocity is less than a , an energy addition tends to increase this velocity, while there is a reduction of the cross-section which acts in the same way, given that the action of these two factors is even more significant as the fluid velocity approaches the value a .

In the region where the fluid velocity is greater than the speed of sound, an energy addition will influence this through a reduction of the fluid velocity, with an increase of the cross-section of the tube through an increase of the fluid velocity.

In any case and from a more general point of view, the velocity and pressure variations occurring for a given flow in a cross-section Σ where the gas attains the conditions W, M, P, T , are the same as for an isentropic flow of a constant composition gas traversing the same cross-section Σ under the same conditions, but for a divergence of the tube (corresponding to this cross-section) equal to:

$$\frac{d\Sigma}{\Sigma} + \frac{dM}{M} - \frac{dQ - dE}{CT}$$

4.2 STEADY STATE CONDITIONS

By application of what has been said with respect to variable state conditions, but this time concerning the special case of steady state conditions, the values of mass-flow μ , cross-section Σ , its gradient $\frac{d\Sigma}{ds}$, the possible heat addition $\frac{dQ}{ds}$, possible variation of chemical potential energy $\frac{d}{ds}$ and possible corresponding variation of mean molecular mass $\frac{dM}{ds}$, may be imposed regardless of the cross-section where the fluid has a velocity different from that of the "speed of sound":

$$a = \sqrt{\gamma \frac{R}{M} T}$$

In the cross-section in question corresponding to the "speed of sound" a , the same quantities on the contrary must satisfy one of the two equivalent relationships:

$$\frac{1}{CT} \frac{dQ - dE}{ds} - \left(\frac{1}{M} \frac{dM}{ds} + \frac{W}{\Sigma} \frac{d\Sigma}{ds} \right) = 0$$

or:

$$\frac{1}{CT} \frac{dQ - dE}{ds} - \left(\frac{1}{M} \frac{dM}{ds} + \frac{1}{\Sigma} \frac{d\Sigma}{ds} \right) = 0,$$

such that the steady state condition can be maintained.

4.2.1 Sign of Velocity Gradient, Location of Throat and Convergence or Divergence of Tube in Sonic Cross-Section

Under steady state conditions, the acceleration of the fluid depends only on the sign of the difference $(a-W)$ and that of the quantity $\left(\frac{dQ - dE}{CT} - \frac{dM}{M} - \frac{d\Sigma}{\Sigma} \right)$.

It is positive if both factors have the same sign, and negative in the opposite case.

In addition, as a consequence of the information discussed in paragraph 4.2 above, we see that the throat of the tube does not correspond to the "sonic" velocity $W=a$ unless the quantity $\left(\frac{dQ - dE}{CT} - \frac{dM}{M} - \frac{d\Sigma}{\Sigma} \right)$

is zero in the cross-section where this velocity is attained, and that the tube on the contrary must present a perfectly defined convergence or divergence $\frac{1}{\Sigma} \frac{d\Sigma}{ds}$ in the cross-section where the velocity W attains

the value a if the quantity in question is different from zero.

Also, as a consequence, the fluid velocity cannot continue to increase beyond the value $W=a$ under steady state conditions unless the quantity $(\frac{dQ-dE}{CT} - \frac{dM}{M} - \frac{d\Sigma}{\Sigma})$ becomes negative in the corresponding cross-sections, such that for a constant cross-section tube, it is necessary to subtract energy from the fluid (or more precisely to make $\frac{dQ-dE}{CT}$ negative) in order to continue to accelerate the fluid beyond the "speed of sound".

4.2.2 Variation of Polytropic Coefficient as Function of W/a

If we except the case of isentropic expansion of a constant composition fluid, and if we assume that it has been arranged such that the fluid velocity is always increasing, its thermodynamic development is characterized under steady state conditions by a local polytropic coefficient n which theoretically deviates from zero for $W=0$ and generally increases with $\frac{W}{a}$.

4.2.3 Variation of Temperature With W/a

Still under steady state conditions, and when the fluid velocity passes through the value $W = \frac{a}{\sqrt{\gamma}}$, the energy flux could be provided to it has no influence on the corresponding local temperature gradient, and this temperature passes through a maximum if the quantity $\frac{dM}{M} + \frac{d\Sigma}{\Sigma}$ is zero.

In any case, the temperature of the fluid passes through a maximum (or a minimum) each time that the condition:

is found to be realized.
$$\frac{M}{R} \left(\frac{a^2}{\gamma W^2} - 1 \right) (dQ - dE) + T \left(\frac{dM}{M} + \frac{d\Sigma}{\Sigma} \right) = 0$$

4.2.4 Detailed Reasons for Special Conditions Required for $W=a$ and Relationships Between $\frac{dW}{W}$, $\frac{d\Sigma}{\Sigma}$ and $\frac{dQ-dE}{CT}$ for $W=a$

In a more general fashion, we may define the physical reasons for which the passage through the critical velocity $a = \sqrt{\frac{\gamma}{\rho}}$ presents special significance in the flow of a fluid by noting that this velocity corresponds to the moment where (as the following calculation demonstrates) the acceleration of the fluid determines, in the absence of any energy addition, identical relative velocity variations and relative specific volume variations; in this case, the variation of the transversal cross-section of the fluid stream (which must be produced so that the passage through the "speed of sound" is possible without discontinuity) depends, for this particular velocity value, only on the possible energy addition and possible variation of mean molecular mass (in fact, these variations being incapable of instantaneously modifying

the acceleration).

In fact, we have seen that for $W=a$, the relative variation of the cross-section must have the value:

$$\frac{d\Sigma}{\Sigma} = \frac{dQ - dE}{CT} - \frac{dM}{M},$$

which can be interpreted by stating that the variation of cross-section which defines the convergence or divergence of the tube in the cross-section where the velocity W attains the value a must be precisely equal in relative value to the variation of specific volume experienced by the fluid in the vicinity of the same cross-section under the influence of the chemical reactions and energy addition to which this is subjected.

In any case, starting with the previously established equations:

$$dP = -\frac{\mu}{\Sigma} dW, \quad \mu = \rho \Sigma W,$$

$$\frac{d\rho}{\rho} + \frac{d\Sigma}{\Sigma} + \frac{dW}{W} = 0$$

and, designating the local polytropic coefficient by n , such that:

$$\frac{dP}{P} = n \frac{d\rho}{\rho},$$

we obviously have:

$$dW = -\frac{\Sigma}{\mu} n P \frac{d\rho}{\rho} = -\frac{\Sigma}{\mu \gamma} n a^2 d\rho,$$

where:

$$\frac{dW}{W} = -\frac{1}{W^2} \frac{n}{\gamma} a^2 \frac{d\rho}{\rho}.$$

Such that by designating the specific volume by v , we can write:

$$\frac{dW}{W} = \frac{n}{\gamma} \frac{a^2}{W^2} \frac{dv}{v},$$

an equation which not only confirms the identity of the relative variations of velocity and specific volume above for $W=a$ and for $n=$, but which demonstrates in addition that the relative velocity variation is greater than the relative volume variation for $W < a$ and lower for $W > a$, with the obvious consequence of the fact that the cross-section inevitably varies in the opposite direction from W , when W is smaller than a , and in the same direction as the velocity when W is larger than a .

In addition, $\frac{dW}{W}$ may be placed into the form:

$$\frac{dW}{W} = \frac{n}{\gamma} \frac{a^2}{W^2} \left(\frac{d\Sigma}{\Sigma} + \frac{dW}{W} \right),$$

where:

$$\boxed{\frac{dW}{W} \left(1 - \frac{n}{\gamma} \frac{a^2}{W^2} \right) = \frac{n}{\gamma} \frac{a^2}{W^2} \frac{d\Sigma}{\Sigma}},$$

an equation which is simplified for $W=a$ to yield:

$$\boxed{\frac{d\Sigma}{\Sigma} = \frac{dW}{W} \left(\frac{\gamma}{n} - 1 \right)}$$

This last equation demonstrates on the one hand that the cross-section Σ must pass through a minimum or maximum such that the velocity gradient remains finite when we have $n = \gamma$ (ie. not only for an isentropic flow, but for any flow fulfilling this condition), and on the other hand that for $n \neq \gamma$, $\frac{dW}{W}$ has a value determined as a function of $\frac{d\Sigma}{\Sigma}$, which then,

as we have seen, must be different from zero, and also exactly equal to the value defined by the condition:

$$\frac{dQ}{CT} - \frac{dE}{M} - \frac{dM}{\Sigma} - \frac{d\Sigma}{\Sigma} = 0$$

which translates the condition from paragraph 4.2 into the case of the steady state condition.

4.3 Speed of Sound

We could not complete the present study without drawing attention to the fact that the quantity $a = \sqrt{\gamma \frac{R}{M} T}$, which we have constantly called the

"speed of sound" according to standard usage, would not actually be the velocity of a small perturbation in the fluid except in the case where this small perturbation would not produce any variation of mean molecular mass in this fluid, and consequently in particular in the case where the fluid would be of constant composition.

On the contrary, if we assume that the fluid composition instantaneously follows the temperature and pressure variations which are imposed on it, such that the chemical or thermodynamic equilibrium is constantly realized, it is obvious that we must consider the fluid flow resulting from the passage of a small perturbation as "isentropic" in the most general meaning of this expression, ie. as corresponding to an approximate entropy (31), or better still a constant total (32). Such flow defines a local polytropic coefficient K at each point, such that:

$$\frac{dP}{P} + K \frac{dv}{v} = 0$$

and it is useless to return to the classical calculations pertaining to the speed of sound to establish that with the hypothesis considered, this velocity has the value:

$$a' = \sqrt{\frac{dP}{d\rho}} = K \frac{P}{\rho}$$

Finally, if the fluid flow resulting from the passage of a small perturbation followed a law intermediate between the two preceding laws,

or even some other law, the same considerations demonstrate that we could still define a local polytropic coefficient K' :

$$K' = \frac{-\frac{dp}{p}}{\frac{dv}{v}},$$

such that the speed of sound would have the value:

$$a'' = K' \frac{p}{\rho}.$$

4.4 Conclusion

In summary, we see that the extension accomplished in the present report of the study method in differential form that we had already applied in 1947 to the special case of steady state flow in cylindrical tubes leads to new general equations [equations (10) and (11) of paragraph 2.2], whose interpretation permits us to recognize immediately for the first time the influence of the principal factors considered.

These equations definitively define the conditions to be fulfilled so that the speed of sound can be exceeded without peculiarity in the flow, as well as the manner in which the fluid flows along its path.

In addition, these two general equations demonstrate the existence in the different cases considered of very surprising details:

-a subtraction of energy at supersonic velocities produces an effect favorable for the acceleration of the fluid.

The velocity and pressure gradients pertaining to a steady state flow with energy addition or chemical reaction are the same in a given cross-section Σ of divergence $\frac{d\Sigma}{\Sigma}$ with steady state flow and isentropic flow of a constant composition gas through the same cross-section, provided that this time the divergence is equal to:

$$\left(\frac{d\Sigma}{\Sigma} + \frac{dM}{M} - \frac{dQ - dE}{CT} \right).$$

For a constant cross-section tube, the exponent n for the polytropic tangent to the fluid flow constantly follows $\gamma \frac{W^2}{a^2}$.

In the general case, the sonic velocity does not correspond to the "throat" of the tube if this "throat" has a precisely sonic velocity.

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